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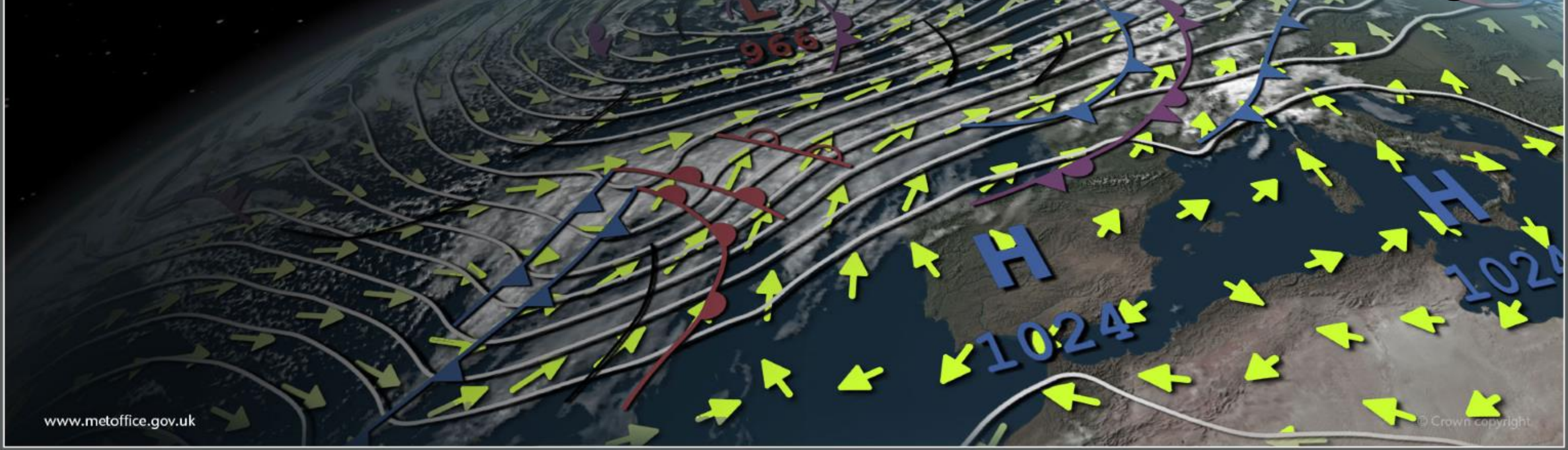
On the diagnosis of model error statistics using weak-constraint data assimilation

Neill Bowler

Thanks to: Mike Cullen, Mihai Popa-Matei, Stephen Pring



Weak-constraint data assimilation

Allow a small correction to the model trajectory every timestep

$$\mathbf{x}_i = M_{i \leftarrow i-1}(\mathbf{x}_{i-1}) + \eta_i$$

Make assumption about correlations in η_i

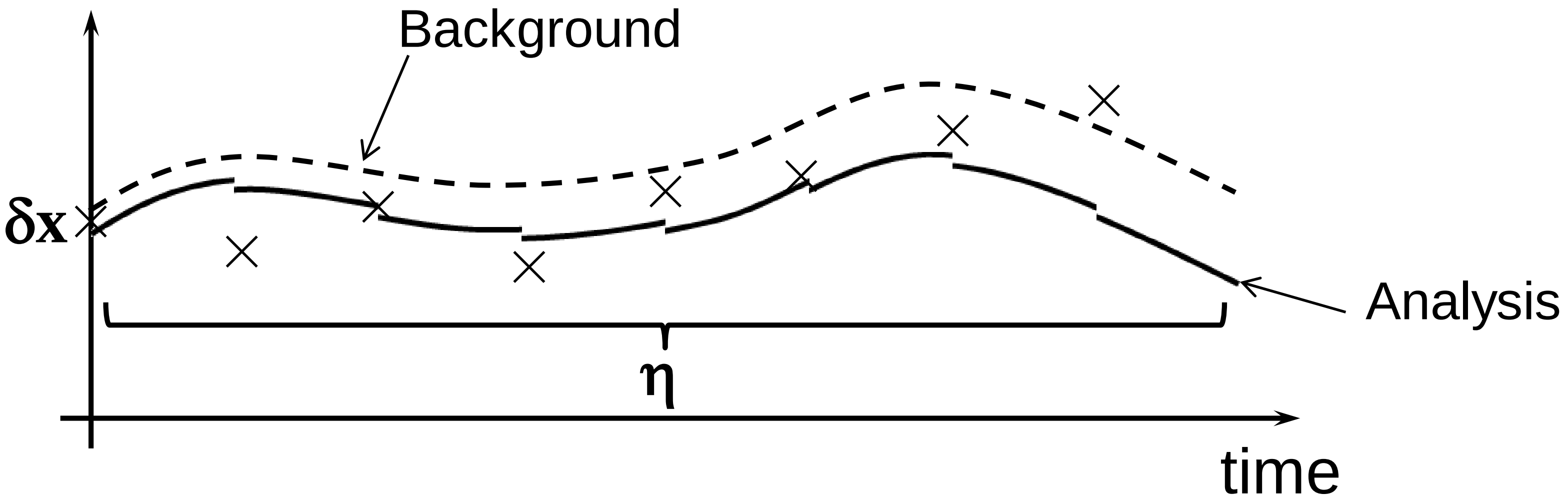
If perfectly correlated

$$\eta_i = \eta / n$$

n – number of timesteps

Other assumptions are available

Weak-constraint data assimilation



Weak-constraint data assimilation

Assume model error is perfectly correlated within DA window

$$J(\delta\mathbf{x},\eta)=\frac{1}{2}\delta\mathbf{x}^T\mathbf{B}^{-1}\delta\mathbf{x}+\frac{1}{2}\eta^T\mathbf{Q}^{-1}\eta$$
$$+\frac{1}{2}\left(\mathbf{y}-H\left(\underline{M}\left(\mathbf{x}_b\right)\right)-\mathbf{H}\underline{\mathbf{M}}\delta\mathbf{x}-\mathbf{H}\underline{\mathbf{N}}\eta\right)^T\mathbf{R}^{-1}\left(\mathbf{y}-H\left(\underline{M}\left(\mathbf{x}_b\right)\right)-\mathbf{H}\underline{\mathbf{M}}\delta\mathbf{x}-\mathbf{H}\underline{\mathbf{N}}\eta\right)$$

$$\underline{M}(\mathbf{x})=\begin{pmatrix} M_{0\leftarrow 0}(\mathbf{x}) \\ M_{1\leftarrow 0}(\mathbf{x}) \\ \vdots \\ M_{n\leftarrow 0}(\mathbf{x}) \end{pmatrix} \qquad \underline{\mathbf{N}}=\frac{1}{n}\begin{pmatrix} 0 \\ \mathbf{M}_{1\leftarrow 1} \\ \mathbf{M}_{2\leftarrow 1}+\mathbf{M}_{2\leftarrow 2} \\ \vdots \\ \sum_{i=1}^{n}\mathbf{M}_{n\leftarrow i} \end{pmatrix}$$

Weak-constraint data assimilation

Assume model error is perfectly correlated within DA window

Using a Kalman filter formulation

$$\delta \mathbf{x} = \mathbf{K}_b \mathbf{d}_b^o \qquad \mathbf{K}_b = \mathbf{B}\underline{\mathbf{M}}^T\mathbf{H}^T\left(\mathbf{R} + \mathbf{H}\underline{\mathbf{M}}\mathbf{B}\underline{\mathbf{M}}^T\mathbf{H}^T + \mathbf{H}\underline{\mathbf{N}}\mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T\right)^{-1}$$

$$\eta = \mathbf{K}_q \mathbf{d}_b^o \qquad \mathbf{K}_q = \mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T\left(\mathbf{R} + \mathbf{H}\underline{\mathbf{M}}\mathbf{B}\underline{\mathbf{M}}^T\mathbf{H}^T + \mathbf{H}\underline{\mathbf{N}}\mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T\right)^{-1}$$

$$\underline{M}(\mathbf{x}) = \begin{pmatrix} M_{0\leftarrow 0}(\mathbf{x}) \\ M_{1\leftarrow 0}(\mathbf{x}) \\ \vdots \\ M_{n\leftarrow 0}(\mathbf{x}) \end{pmatrix} \qquad \underline{\mathbf{N}} = \frac{1}{n}\begin{pmatrix} 0 \\ \mathbf{M}_{1\leftarrow 1} \\ \mathbf{M}_{2\leftarrow 1} + \mathbf{M}_{2\leftarrow 2} \\ \vdots \\ \sum_{i=1}^{n} \mathbf{M}_{n\leftarrow i} \end{pmatrix}$$

Desroziers Diagnostics

A method for estimating covariance matrices using analysis increments and innovations

Can be used to diagnose **R**, **B** and **A** in observation space – otherwise poorly known

Information out of DA – being used to diagnose observation errors

Warning: Separate diagnoses are not independent and errors in one matrix can spread – prior information is required

Model Error Diagnostics

The innovation is defined by

$$\mathbf{d}_b^o = \mathbf{y} - H\left(\underline{M}\left(\mathbf{x}_b\right)\right)$$

$$\mathbf{d}_b^o = \mathbf{y} - H\left(\mathbf{x}_t\right) - \left[H\left(\underline{M}\left(\mathbf{x}_b\right)\right) - H\left(\mathbf{x}_t\right)\right]$$

$$\mathbf{d}_b^o \cong \varepsilon_o - \mathbf{H}\underline{\mathbf{M}}\varepsilon_b - \mathbf{H}\underline{\mathbf{N}}\varepsilon_q$$

Innovation covariance

$$E\left(\mathbf{d}_b^o\left(\mathbf{d}_b^o\right)^T\right) \cong E\left(\varepsilon_o\left(\varepsilon_o\right)^T\right) + \mathbf{H}\underline{\mathbf{M}}E\left(\varepsilon_b\left(\varepsilon_b\right)^T\right)\underline{\mathbf{M}}^T\mathbf{H}^T + \mathbf{H}\underline{\mathbf{N}}E\left(\varepsilon_q\left(\varepsilon_q\right)^T\right)\underline{\mathbf{N}}^T\mathbf{H}^T$$

$$E\left(\mathbf{d}_b^o\left(\mathbf{d}_b^o\right)^T\right) \cong \mathbf{R} + \mathbf{H}\underline{\mathbf{M}}\mathbf{B}\underline{\mathbf{M}}^T\mathbf{H}^T + \mathbf{H}\underline{\mathbf{N}}\mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T$$

Model Error Diagnostics

Model forcing increments

$$\eta = \mathbf{K}_q \mathbf{d}_b^o$$

So, the cross-product with the innovations

$$E\left(\mathbf{H}\underline{\mathbf{N}}\eta\left(\mathbf{d}_b^o\right)^T\right) \cong \mathbf{H}\underline{\mathbf{N}}\mathbf{K}_q E\left(\mathbf{d}_b^o\left(\mathbf{d}_b^o\right)^T\right)$$

$$E\left(\mathbf{H}\underline{\mathbf{N}}\eta\left(\mathbf{d}_b^o\right)^T\right) \cong \mathbf{H}\underline{\mathbf{N}}\mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T\left(\mathbf{R} + \mathbf{H}\underline{\mathbf{M}}\mathbf{B}\underline{\mathbf{M}}^T\mathbf{H}^T + \mathbf{H}\underline{\mathbf{N}}\mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T\right)^{-1} E\left(\mathbf{d}_b^o\left(\mathbf{d}_b^o\right)^T\right)$$

$$E\left(\mathbf{H}\underline{\mathbf{N}}\eta\left(\mathbf{d}_b^o\right)^T\right) \cong \mathbf{H}\underline{\mathbf{N}}\mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T$$

This in not a covariance matrix

Model error diagnostics

How has separation been achieved?

Initial condition errors at start

Model errors throughout the window

η-> constant

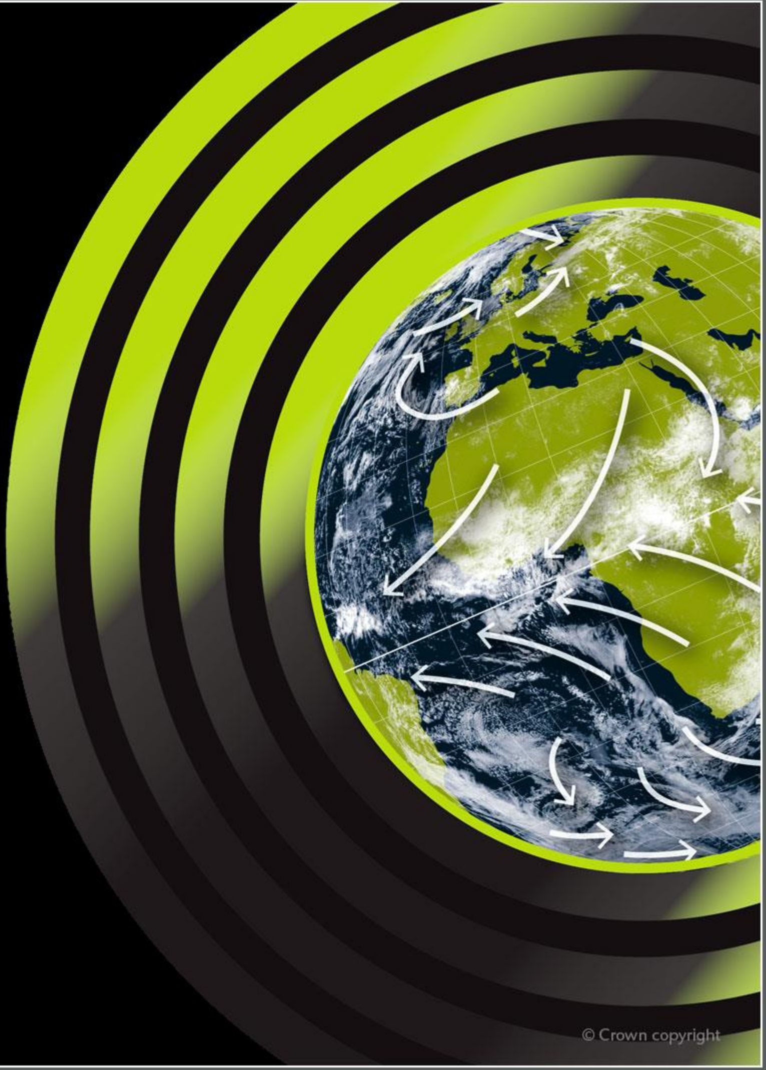
Correction term $\left(\mathbf{R}+\mathbf{H}\underline{\mathbf{M}}\mathbf{B}\underline{\mathbf{M}}^T\mathbf{H}^T+\mathbf{H}\underline{\mathbf{N}}\mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T\right)^{-1}E\left(\mathbf{d}_b^o\left(\mathbf{d}_b^o\right)^T\right)$

Note that **B, Q** are not available, only $\mathbf{H}\underline{\mathbf{M}}\mathbf{B}\underline{\mathbf{M}}^T\mathbf{H}^T$, $\mathbf{H}\underline{\mathbf{N}}\mathbf{Q}\underline{\mathbf{N}}^T\mathbf{H}^T$

B – observations from start of window

Q – observations at time 1

Simple model tests



Lorenz 95

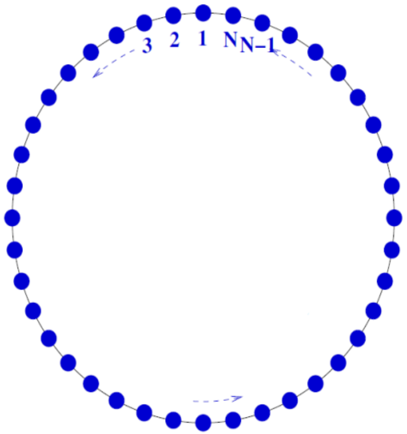
1d model with variables defined on a latitude circle

$$\frac{dx_i}{dt} = x_{i-1}\left(x_{i+1} - x_{i-2}\right) - x_i + F$$

i=1,2, ..., N cyclic boundary conditions with N=40 and F=8

Allegorical of NWP models with Δt=1 associated with 5 days

Integration with Euler method with time-step Δt=0.005



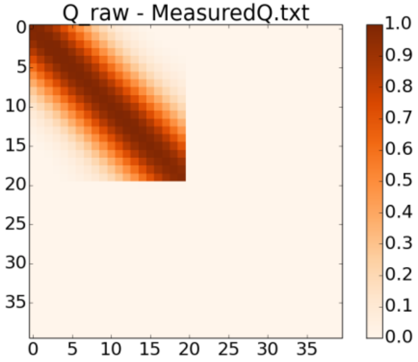
Lorenz 95 imperfect

Each TS, perturbations added with correlation length of 5 grid-points

Perturbations constant over DA window

Perturbations restricted to first half of the grid

Perturbations applied to truth model



$$\frac{dx_i}{dt} = x_{i-1}\left(x_{i+1} - x_{i-2}\right) - x_i + F + P$$

Lorenz 2005, model 2

Based on the Lorenz 95 model, but introduces spatial correlations

$$\frac{dx_i}{dt} = [x,x]_{K,i} - x_i + F$$

$$W_i = \sum_{j=-J}^{J}{}' \frac{x_{i-j}}{K}$$

$$[x,x]_{K,i} = -W_{i-2K}W_{i-K} + \sum_{j=-J}^{J}{}' \frac{W_{i-K+j}x_{i+k+j}}{K}$$

$$N = 120$$
$$J = \mathrm{int}(K/2)$$
$$K = 6$$

Σ' is a modified summation (if K is even, give half weight to the end points)

Integration with Euler method with time-step Δt=0.005

All variables observed every 10Δt

Data assimilation window uses observations from 3 times (window length Δt=0.1)

Lorenz 2005, model 2, imperfect

Impose an error in the modified summation

For the forecast model, divide the end points by 1.02

$$W_i = \frac{\frac{1}{2}x_{i-3} + x_{i-2} + x_{i-1} + x_i + x_{i+1} + x_{i+2} + \frac{1}{2}x_{i+3}}{6}$$

$$\longrightarrow \quad W_i = \frac{\frac{1}{2.04}x_{i-3} + x_{i-2} + x_{i-1} + x_i + x_{i+1} + x_{i+2} + \frac{1}{2.04}x_{i+3}}{6}$$

Applies to all the modified summations in the model

Data assimilation settings

Weak-constraint 4DVar

All variables observed every $10\Delta t=0.05$

Data assimilation window uses observations from 3 times (window length $\Delta t=0.1$)

Static covariance matrices are used

Initial $\mathbf{B}$ and $\mathbf{Q}$ covariance matrices similar, and far from ideal

Observations – perturb truth, variance 0.01

Initial $\mathbf{R}$ matrix is the true covariance matrix, and not cycled

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Lorenz '95 results

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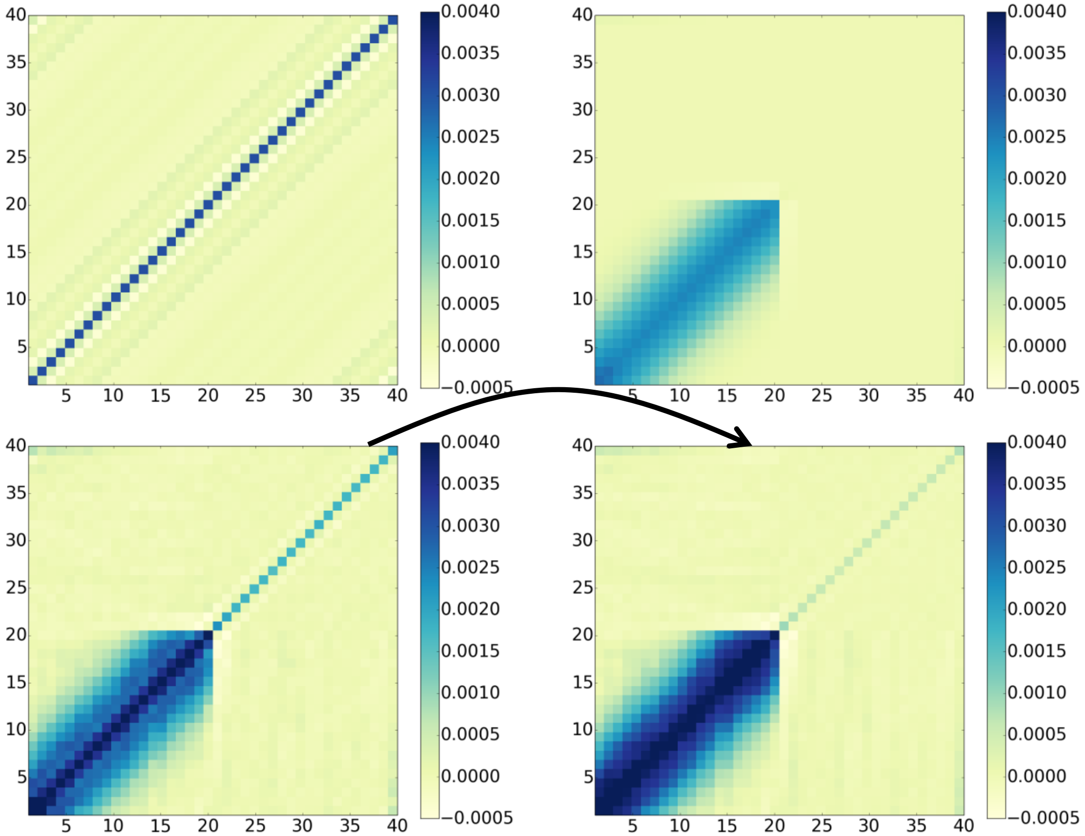
Lorenz 95 $\mathbf{Q}$

Input matrix

True model errors



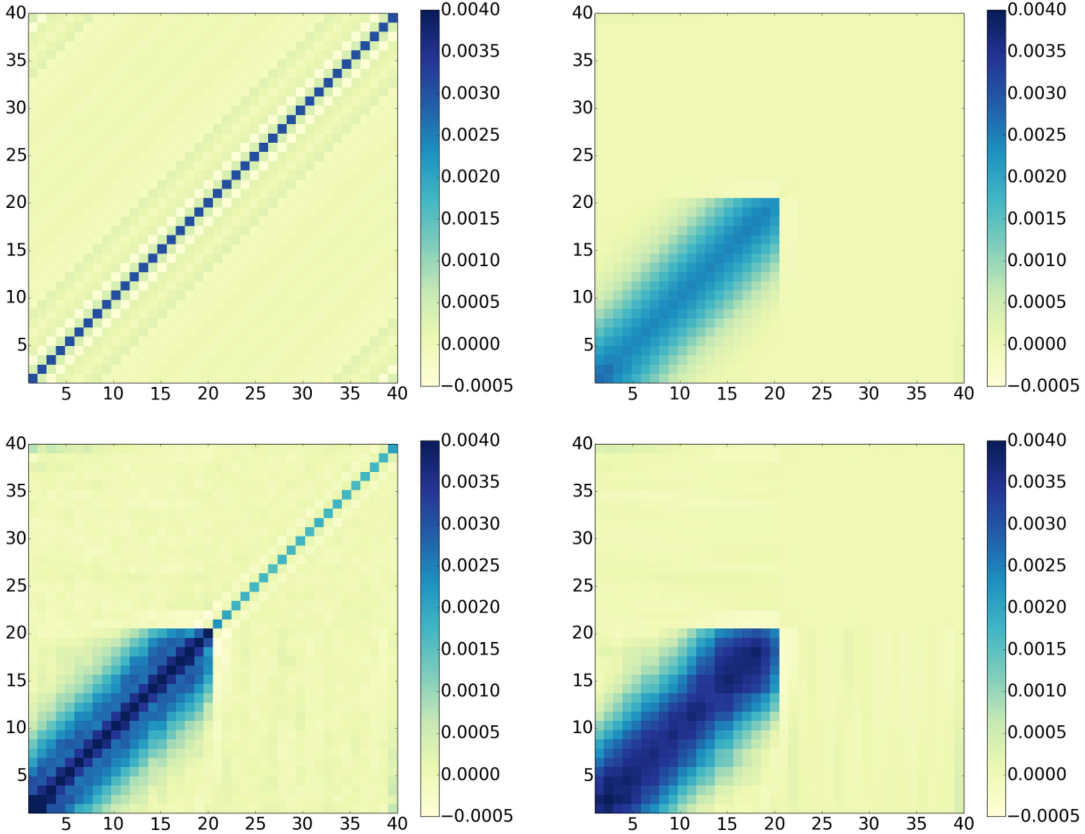
1st estimate

2nd estimate

Lorenz 95 $\mathbf{Q}$

Input matrix

True model errors

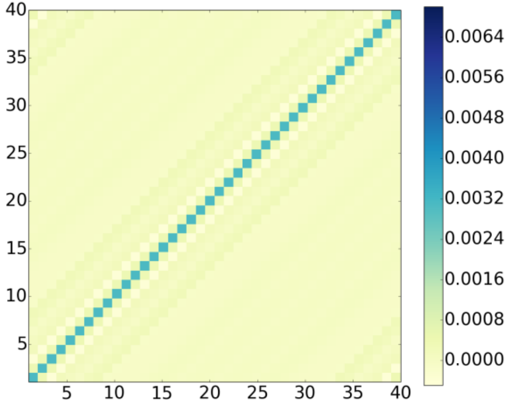


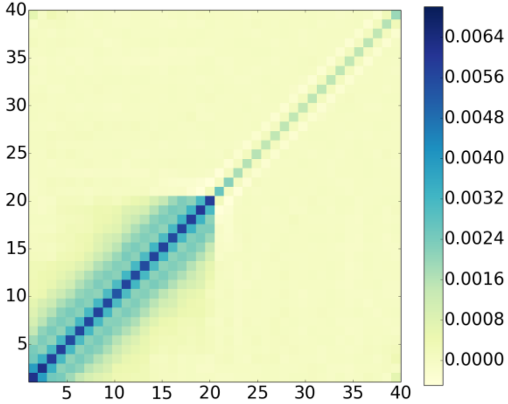
1st estimate

10th estimate

Lorenz 95 $\mathbf{B}$ – initial time obs

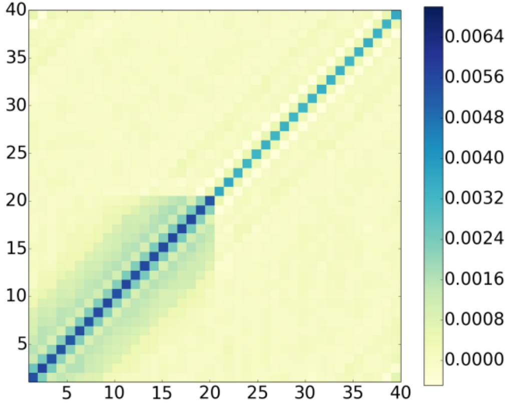
Input matrix

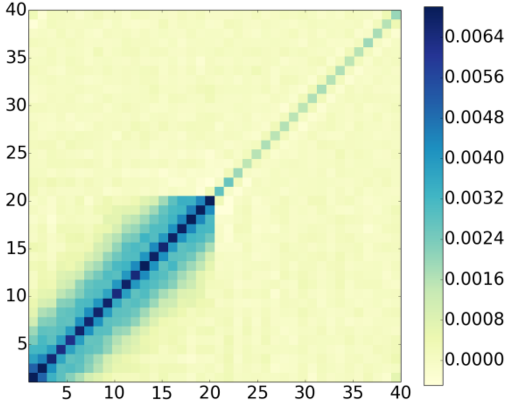




True background errors

1st estimate



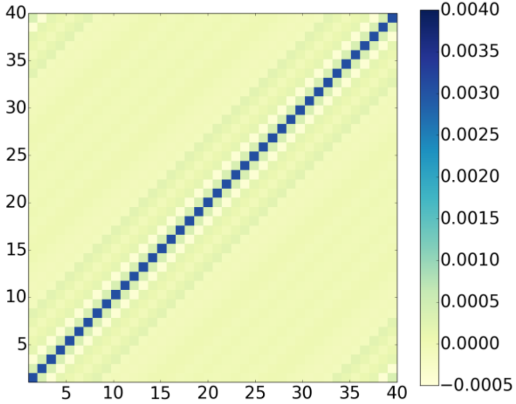


10th estimate

Lorenz ‘95 results – no error

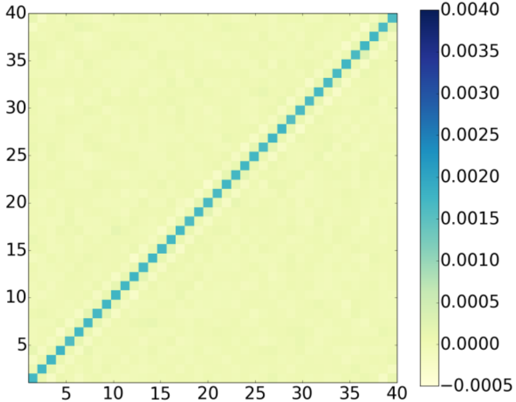
Lorenz 95 $\mathbf{Q}$

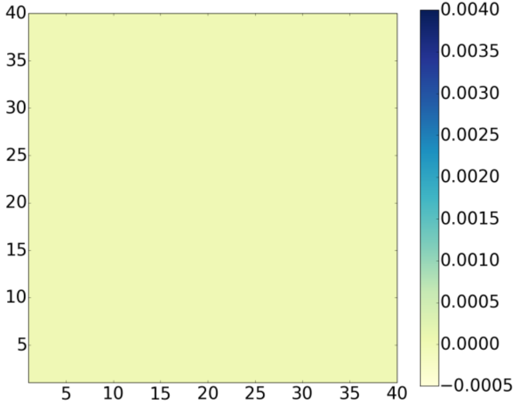
Input matrix



True model errors

1st estimate





10th estimate

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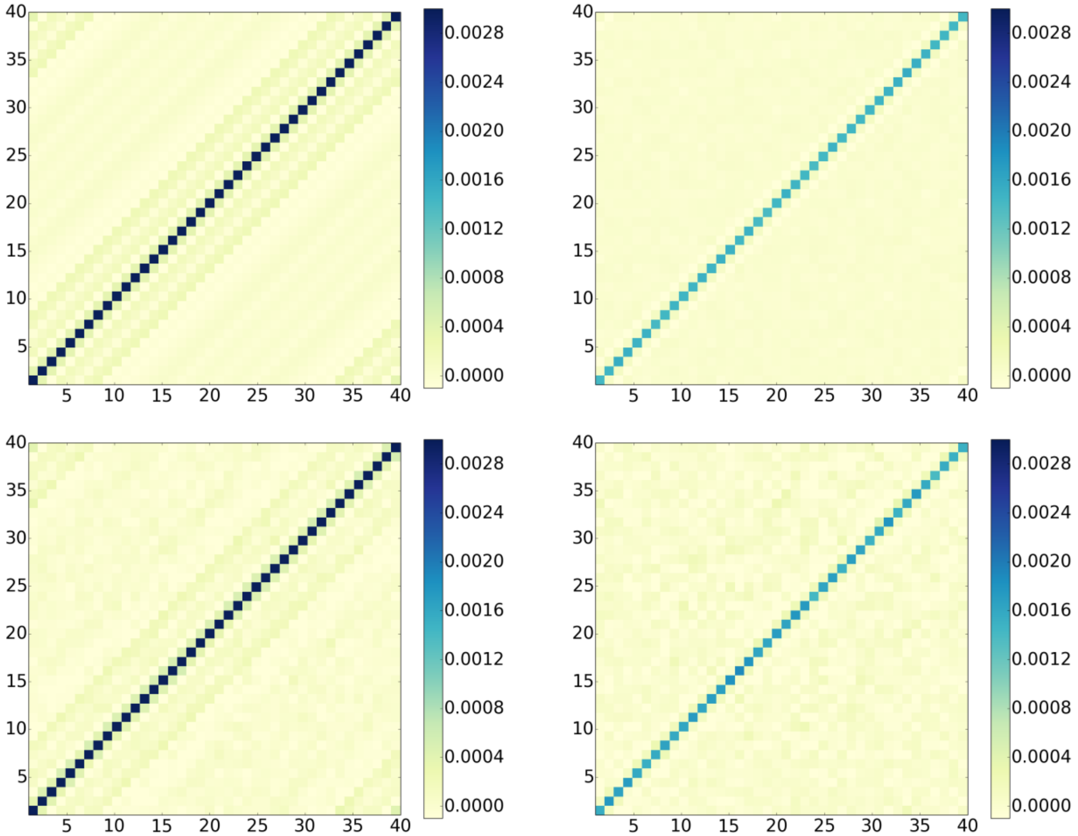
Lorenz 95 $\mathbf{B}$

Input matrix

True background errors

1st estimate

10th estimate



Lorenz ‘05 results

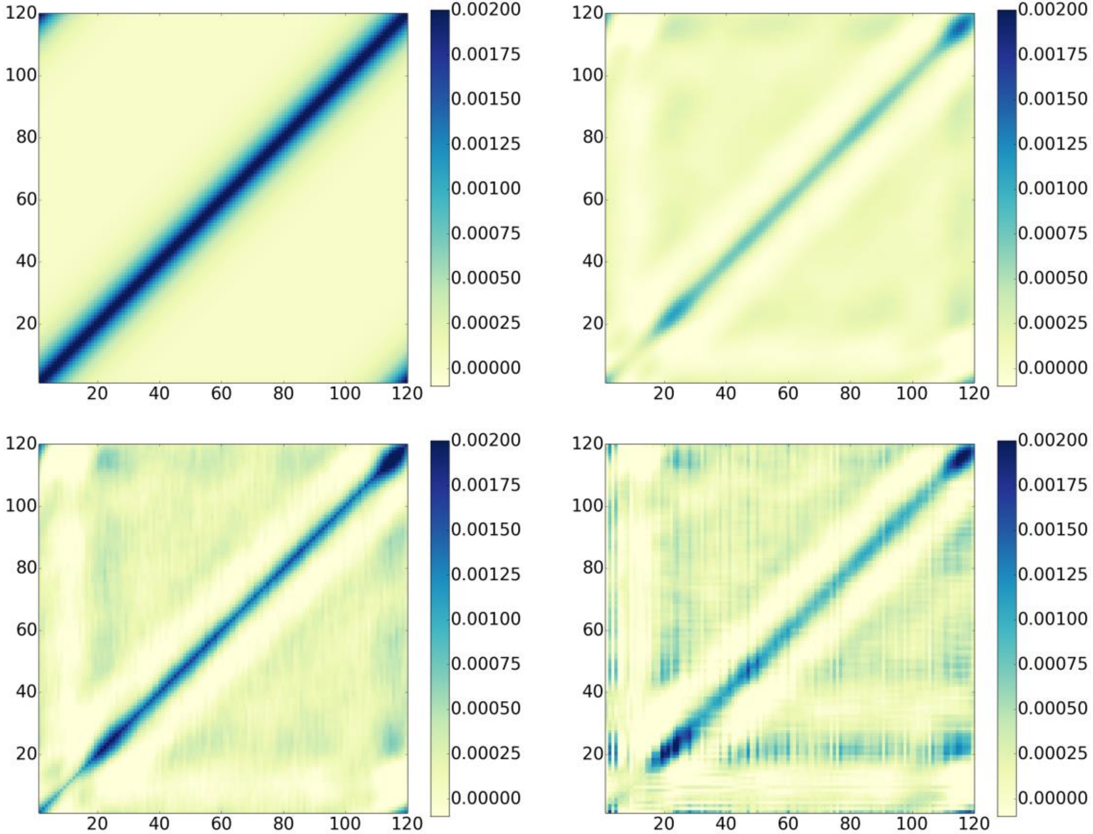
Lorenz 05 $\mathbf{Q}$

Input matrix

True model errors

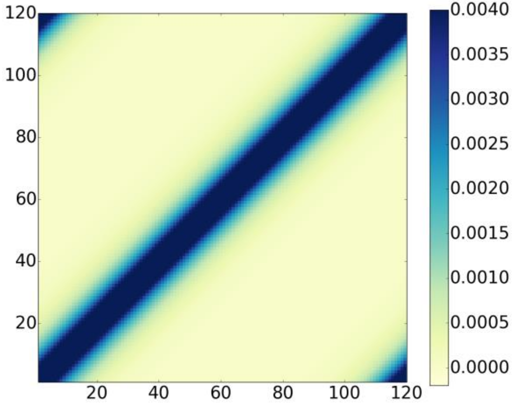
1st estimate

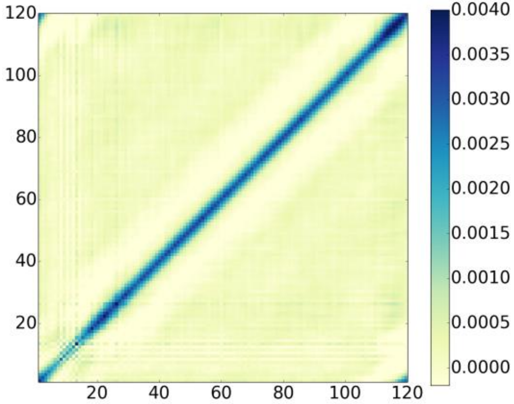
10th estimate



Lorenz 05 **B**

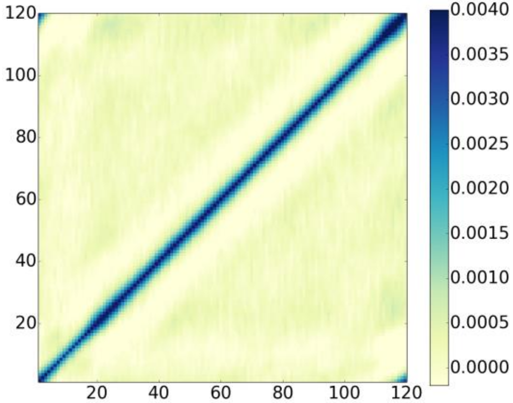
Input matrix

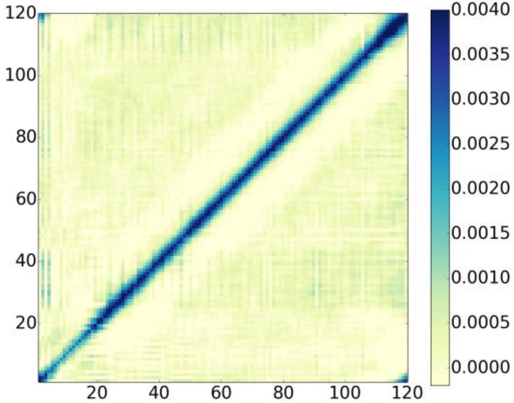




True background errors

1^{st} estimate





10^{th} estimate

Comparisons

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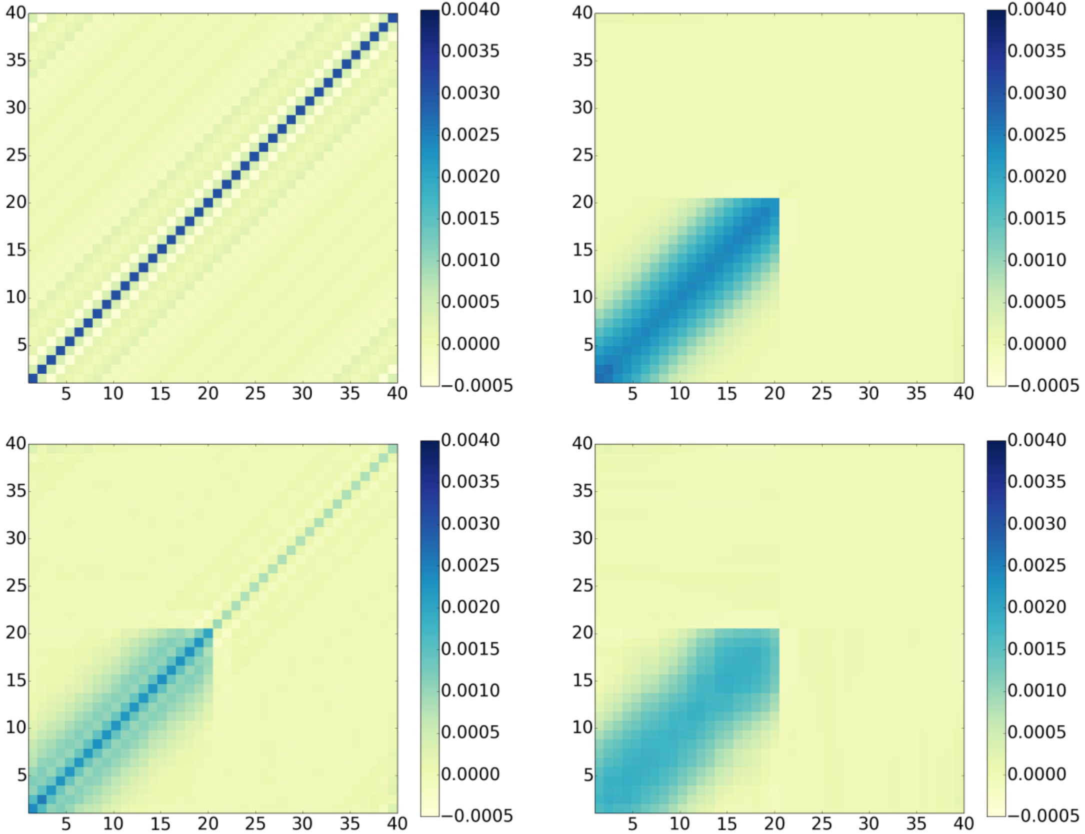
Comparison with analysis increment covariance

Input matrix

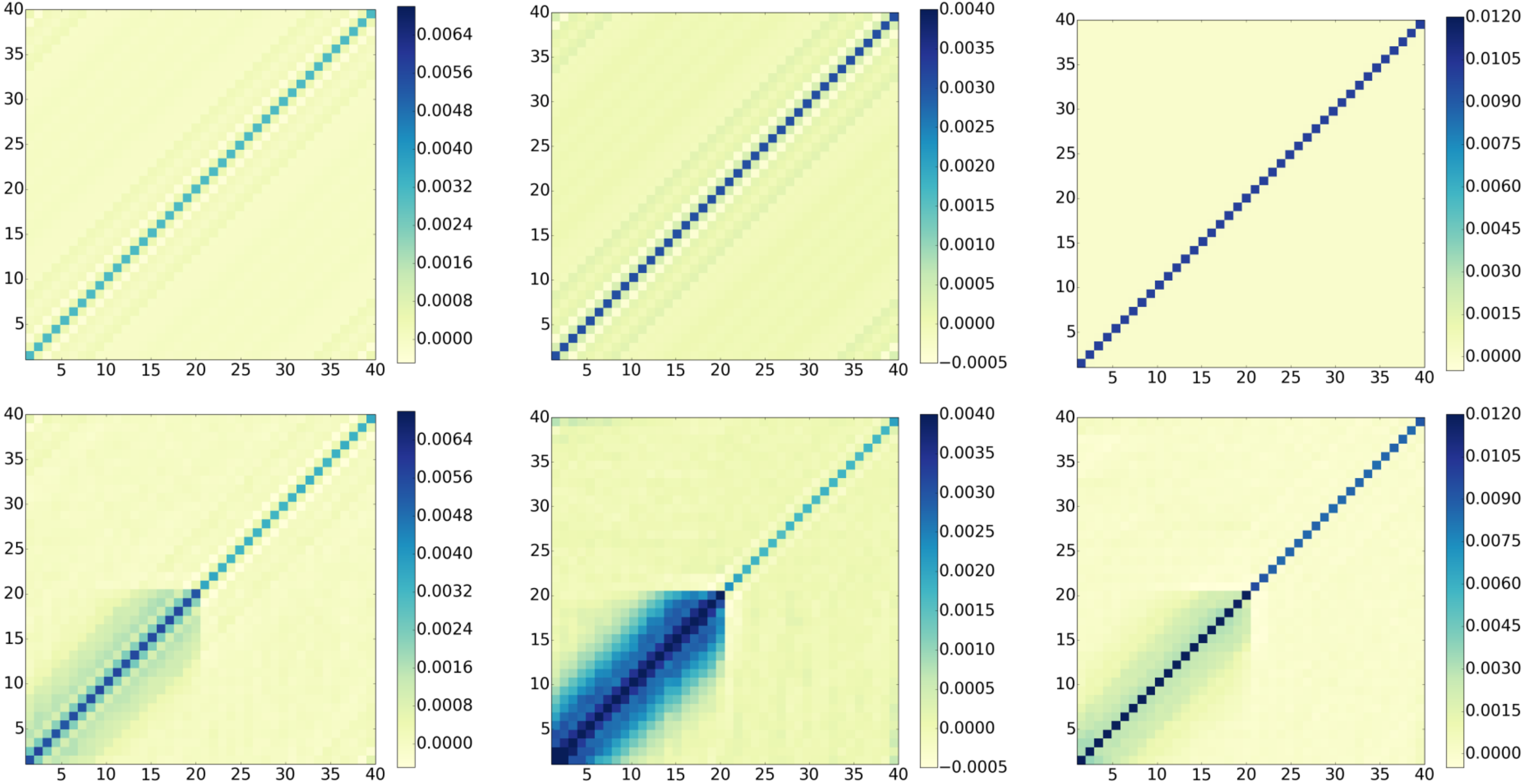
True model errors

1st estimate

10th estimate



Estimating **B**, **Q** and **R**



Input Matrix

1^{st} estimate

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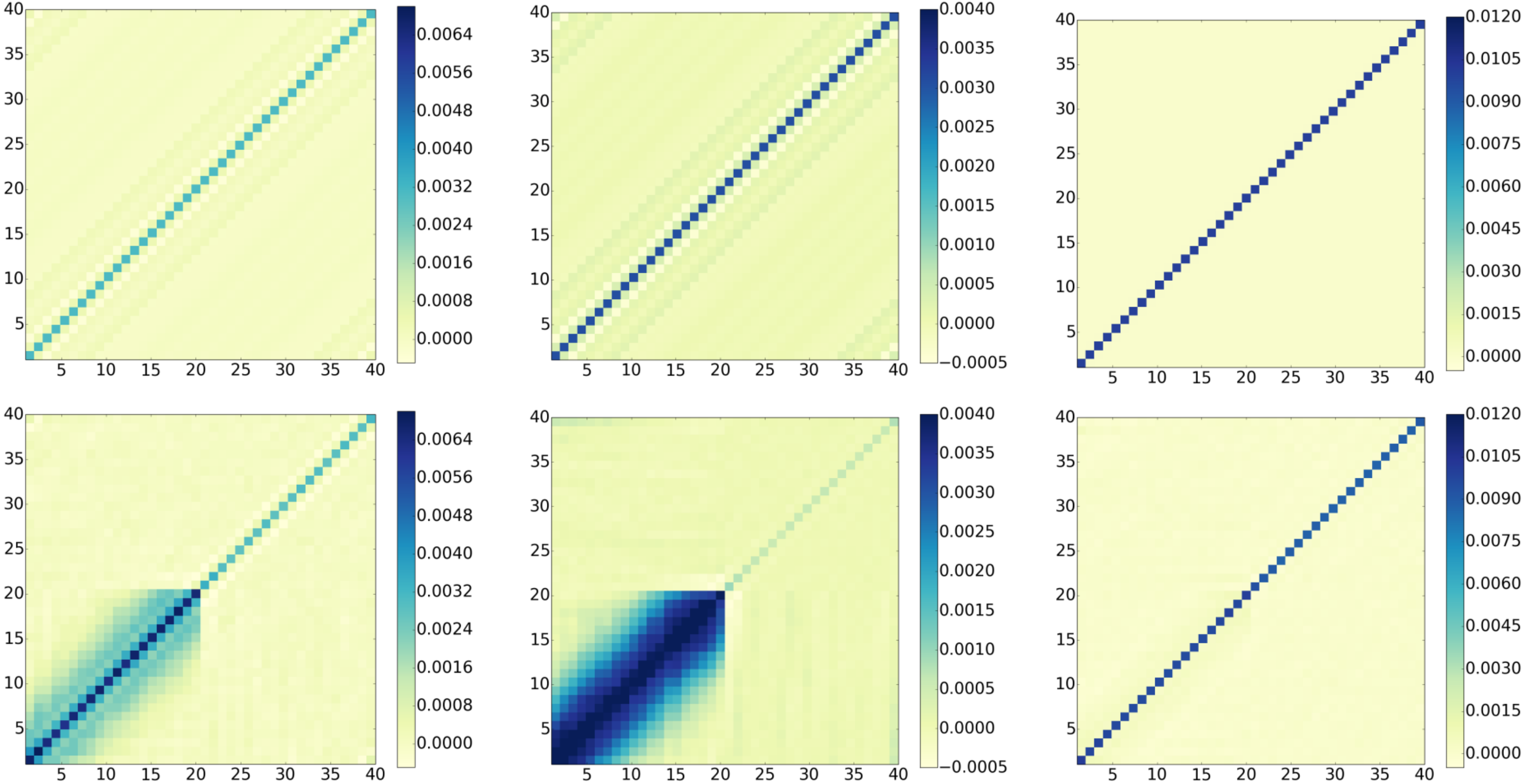
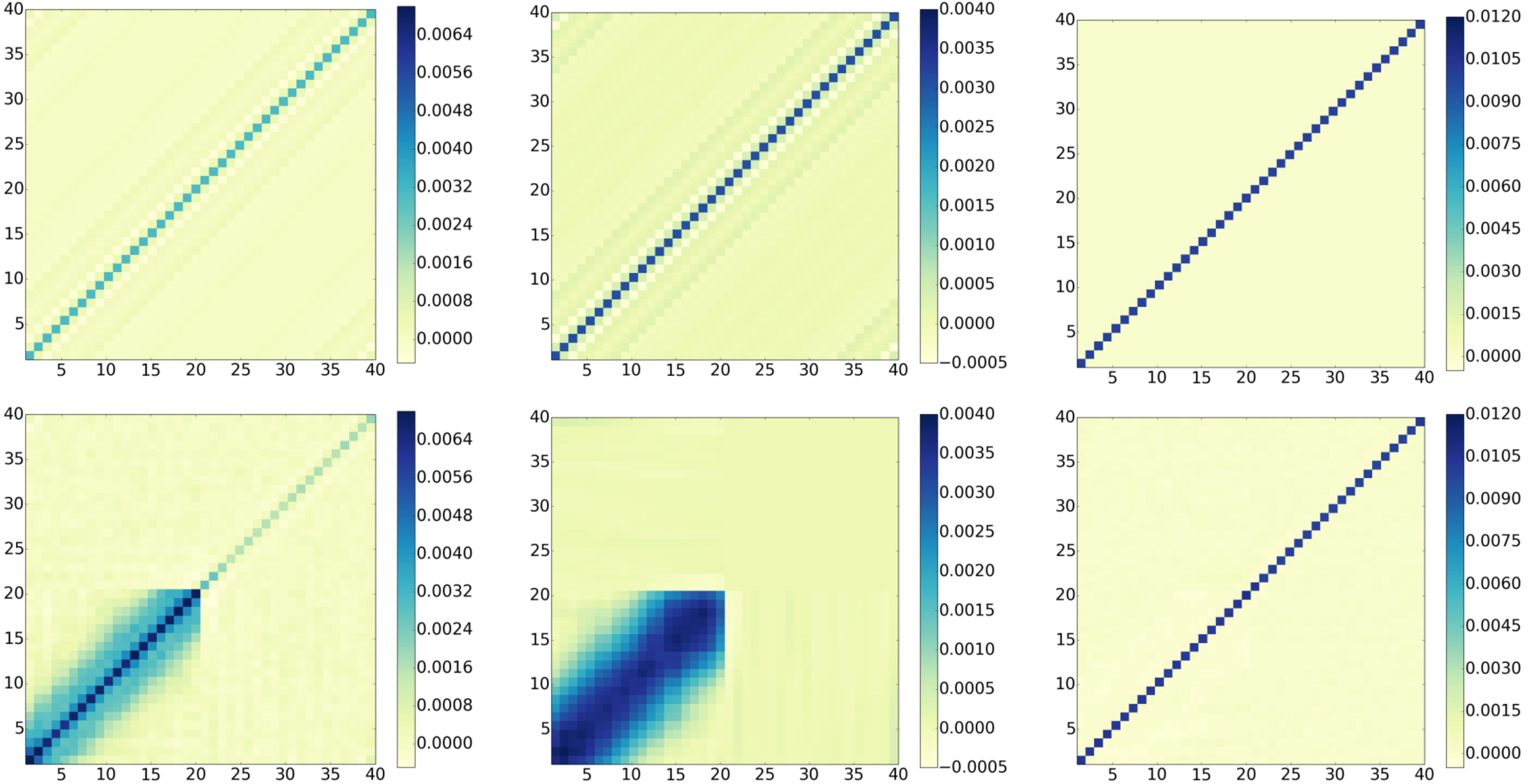
Estimating **B**, **Q** and **R**



Input Matrix

2nd estimate

Estimating **B**, **Q** and **R**



Input Matrix

10^{th} estimate

Summary

Successfully separated model, background and observation error by extending method of Desroziers

- Both when model error is present, and when it is not

Assumptions about model error are important

- Separation less successful if model error uncorrelated

Initial estimate of **Q** -> Weak-constraint DA -> Revised **Q**

- Requires a model for model error
- Uses assumptions built into DA

Limitations

Statistical method – estimate parameters of a stationary model

Here: model errors are state independent

Desroziers diagnostics are available in observation space

Prior knowledge needed to separate the matrices

Assumes background, model and observation errors uncorrelated

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Thank you

References

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