Prediction and verification of extremes

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What are Extremes?

- **Weather and Climate:**
  Rare, exceptional, "big" and potential of **high impact**
  complex and multivariate phenomena

- **Mathematically:**
  Block maxima or exceedances over high threshold
  Events in tail of distribution

Snow storm Münsterland November 2005
Harald Schmidt Show

The verification problem

- Small number of observed events
- Weak representation of extremes in models – calibration
- Standard verification measures degenerate as event rarity increases
- Large uncertainties in predictions and verification measures

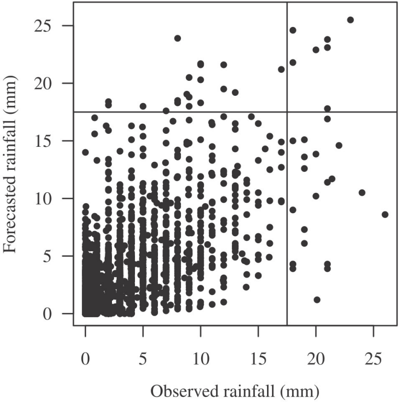
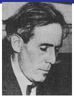


FIG. 1. Forecasted 6-h rainfall accumulations against observations at Eskdalemuir.

From Ferro and Stephenson (2011)

Outline

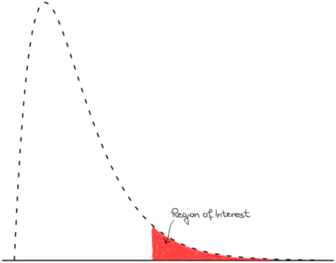
1. Extreme value theory
   - a Univariate extreme value theory
   - b Bivariate extremes
2. Deterministic prediction/verification
   - a Contingency table
   - b Extreme value model
   - c Extremal dependence indices
3. Probabilistic prediction/verification
   - a Proper scoring rules
   - b Proper scoring rules for extremes
   - c Downscaling of precipitation extremes



"Il est impossible que l'improbable n'arrive jamais"

Emil Julius Gumbel (1891-1966)

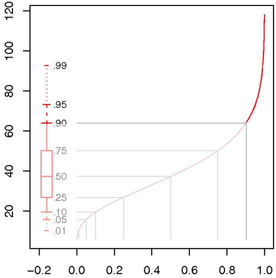
- Extremal indices
  95% quantile of daily precipitation:
  9mm im winter, 15mm summer
- Not really extreme!
- Extreme: Q100



"Going beyond the range of the data" (Philippe Naveau)

Probabilistic concept on asymptotic behaviour of extremes

Extreme value theory "Going beyond the range of the data"



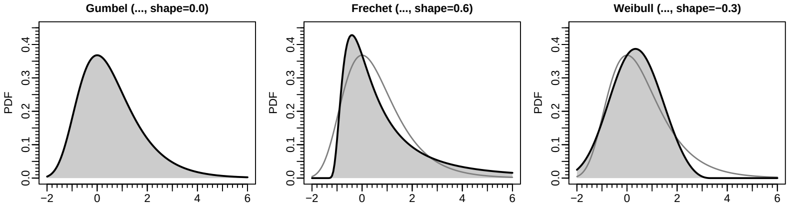
- **Limit theorem** for sample maxima
  → asymptotic distribution of extremes
- Condition of **max-stability** (de Haan, 1984)
  → Maxima follow a generalized extreme value distribution (GEV)
- **universal behaviour of extremes**
  → allows for extrapolation!

In praxis not enough – not in asymptotitic limit – bad convergence

Generalised extreme value distribution (GEV)

Maxima of large samples $X_N = \max\{z^{(1)}, \ldots, z^{(N)}\}$ asymptotically follow $N \to \infty$ a GEV

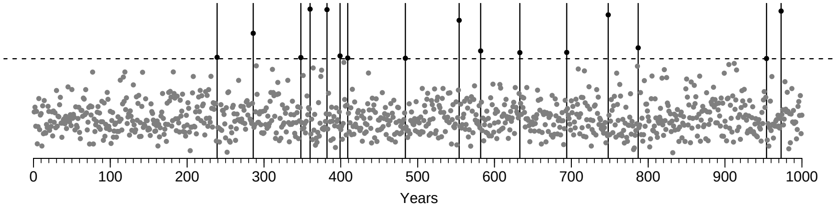
$$G_X(x) = \begin{cases} \exp(-(1 + \xi \frac{x-\mu}{\sigma})^{-1/\xi})_+, & \xi \neq 0 \\ \exp(-\exp(-\frac{xfor-\mu}{\sigma})), & \xi = 0 \end{cases},$$



Generalised Pareto distribution (GPD)

Analogue to GEV, but for peaks-over-threshold (POT) $Y = X - u$ asymptotically for $u \to \infty$ follow a GPD

$$H_Y(y;u) = \begin{cases} 1 - \left(1 + \xi \frac{y}{\sigma_u}\right)^{-1/\xi}, & \xi \neq 0 \\ 1 - \exp(\frac{y}{\sigma_u}), & \xi = 0 \end{cases},$$



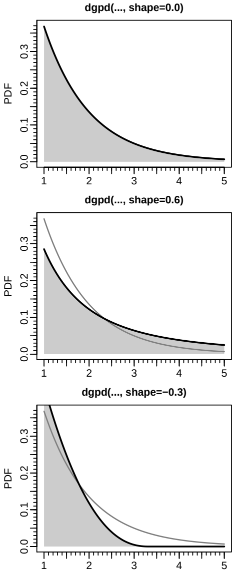
Generalised Pareto distribiton (GPD)

GPD class:

$\xi = 0$ Exponential distribution (Gumbel type)

$\xi > 0$ Pareto tail (Fréchet type)

$\xi < 0$ Beta (Weibull type)



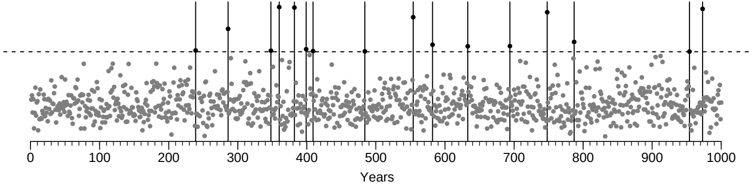
Poisson point process

For high theshold u, $X_i > u$ is asymptotically a Poisson point process on $[0,1] \times (u, \infty)$ with intensity

$$\Lambda(A) = (t_2 - t_1)\left(1 + \xi\left(\frac{y-\mu}{\sigma}\right)\right)^{-1/\xi}.$$

for $A = [t_1, t_2] \times (u, z)$

μ, σ and ξ parameter of corresponding GEV



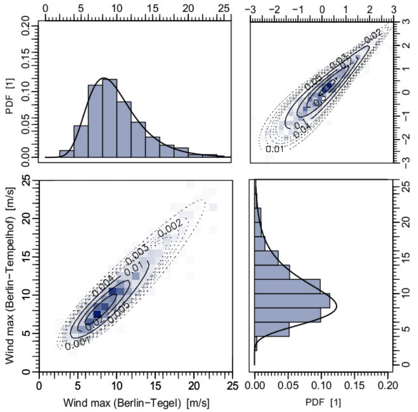
Multivariate extreme value statistics

Marginal distribution:

- Standard Fréchet or Gumbel

Dependence structure:

- Limit theorem for multivariate sample maxima
- Max-stability for marginals and dependence structure



From Schoelzel and Friederichs (2009)

Bivariate EVD – dependence structure

$$Pr(X \leq x, Y \leq y) = \exp\{-V(x,y)\}$$

$$\chi = \lim_{z\to\infty} Pr(X > z | Y > z)$$

- $\chi \neq 0$ asymptotic dependence
- $\chi = 0$ asymptotic independence

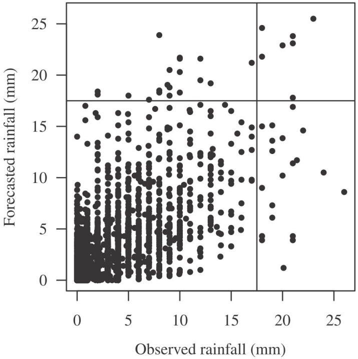


FIG. 1. Forecasted 6-h rainfall accumulations against observations at Eskdalemuir.

From Ferro and Stephenson (2011)

1. Conclusions

- Extreme value theory – universal law for extreme values
  - Univariate – GEV or GPD; Poisson point process
  - Multivariate – Marginals and dependence structure
- Max-stability – characteristics of extremes unchanged
  - Tail behaviour – shape parameter ξ
  - Dependence structure – asymptotic dependence or independence

Extrapolation to really extreme values

Contingency table for infinite sample

	event observed	non-event observed	
event forecasted	$Pr(X > x, Y > y) = Hp$	$Pr(X \leq x, Y > y) = F(1-p)$	$Pr(Y > y)$
non-event forecasted	$Pr(X > x, Y \leq y) = (1-H)p$	$Pr(X \leq x, Y \leq y) = (1-F)(1-p)$	$Pr(Y \leq y)$
	$Pr(X > x) = p$ (baserate)	$Pr(X \leq x) = 1-p$	

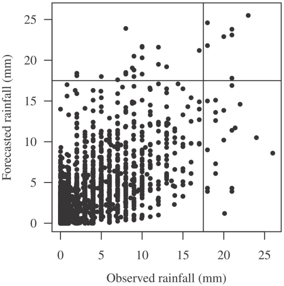


FIG. 1. Forecasted 6-h rainfall accumulations against observations

Extreme value model

Transform bivariate random variable

$$\tilde{X} = -\log(1 - F_X(X)) \qquad \tilde{Y} = -\log(1 - F_Y(Y)),$$

u and v are $(1-p)$-quantiles – $F_X(u) = 1 - p$ and $F_Y(v) = 1 - p$

Let $Z = \min\{\tilde{X}, \tilde{Y}\}$

$$Pr(X > u, Y > v) = Pr\left(Z > -\log p\right)$$

Extreme value theory (Ledford, Tawn, 1996; Ferro, 2007)

$$Pr(Z > -\log p) = \kappa p^{1/\eta}. \tag{1}$$

Contingency table for extreme value model

Ferro (2007)

	event observed	non-event observed	
event forecasted	$Pr(X > u, Y > v)$ $= \kappa p^{1/\eta}$	$Pr(X \leq u, Y > v)$ $= p - \kappa p^{1/\eta}$	p
non-event forecasted	$Pr(X > u, Y \leq v)$ $= p - \kappa p^{1/\eta}$	$Pr(X \leq u, Y \leq v)$ $= 1 - 2p + \kappa p^{1/\eta}$	$1 - p$
	p	$1 - p$	

Contingency table for extreme value model

Maximum likelihood estimators

$$\hat{\eta} = \min\{1, \frac{1}{m} \sum_{t:z_t>w_0} (z_t - w_0)\},$$

m is number of $z_t > w_0$

$$\hat{\kappa} = \frac{m}{n} \exp\left(\frac{w_0}{\hat{\eta}}\right).$$

$w_0 < p$, but large enough for extreme value model to be valid

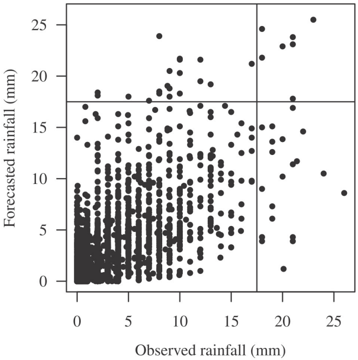


FIG. 1. Forecasted 6-h rainfall accumulations against observations at Eskdalemuir.

From Ferro and Stephenson (2011)

Extremal dependence indices

Ferro and Stephenson (2011) propose EDI and SEDI

$$\mathrm{EDI} = \frac{\log F - \log H}{\log F + \log H}$$

$$\mathrm{SEDI} = \frac{\log F - \log(1-F) - \log H + \log(1-H)}{\log F + \log H + \log(1-F) + \log(1-H)}$$

both of which are equitable, independent of baseline p, do have non-degenerate limits

2. Conclusions

- Contingency table for extreme value model
- EDI and SEDI – scores for extreme events

Probabilistic prediction

- Rarity of events – probabilistic prediction
- Provides more information for decision makers
- Predictive distribution $F \in \mathcal{F}$
- Observation $Y \in \Omega_Y$

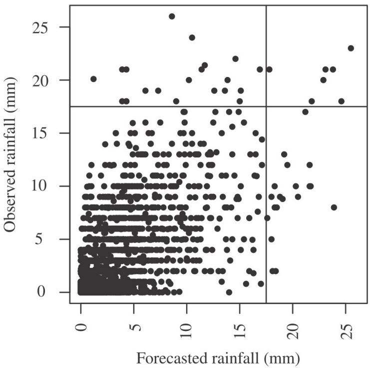


FIG. 1. Forecasted 6-h rainfall accumulations against observations at Eskdalemuir.

From Ferro and Stephenson (2011)

Proper scoring rules

According to Gneiting and Raftery (2007):

A Score function is **proper** if

$$E_F[S(F,Y)] = \int_{\Omega_Y} S(F,y)dF(y) \leq E_F[S(G,Y)], \quad F,G \in \mathcal{F}$$

and **strictly proper**

$E_F[S(F,Y)] = E_F[S(G,Y)]$ only if $F = G$

Continuous ranked probability score

$$
\begin{aligned}
\mathrm{CRPS}(F, y) &= \int_{-\infty}^{\infty} (F(t) - I_{y \leq y})^2 \, dt \\
&= \int S_{BS}(y, F_Y, u) du \quad \text{Brier score} \\
&= 2 \int_0^1 (I_{y \leq F^{-1}(\tau)} - \tau)(F^{-1}(\tau) - y) d\tau \\
&= 2 \int_0^1 S_{QS}(y, F_Y, \tau) d\tau \quad \text{Quantile score}
\end{aligned}
$$

Logarithmic score

$$
\mathrm{LogS}(F, y) = -\log f(y), \quad f(y) = F'(y),
$$

Proper scoring rules – focus on extremes

WRONG!: Conditioning verification on a subset of observations
Lerch et al. (2015)

CORRECT!: Stratify with respect to forecasts
Gneiting and Ranjan (2011); Diks et al. (2011)

Weighted CRPS

Gneiting, Ranjan (2011)

Threshold-weighted CRPS

$$\mathrm{CRPS}^{u}(F,y) = \int_{-\infty}^{\infty} \left(F(t) - I_{y\leq y}\right)^2 u(t)dt$$

Quantile-weighted CRPS

$$\mathrm{CRPS}^{q}(F,y) = 2\int_{0}^{1} (I_{y\leq F^{-1}(\tau)} - \tau)(F^{-1}(\tau) - y)q(\tau)d\tau$$

Conditional and censored likelihood score

Diks et al. (2011)

Conditional likelihood score

$$\mathrm{CL}(F, y) = -I_{y \in A} \log \left(\frac{f(y)}{\int_A f(s) ds} \right)$$

Censored likelihood score

$$\mathrm{CSL}(F, y) = -\left(I_{y \in A} \log f(y) + I_{y \in A^C} \log \left(\int_{A^C} f(s) ds \right) \right)$$

Non-stationary Poisson point process

Friederichs (2010)

Intensity

$$\Lambda(A) = (t_2 - t_1)\left(1 + \xi\left(\frac{y-\mu}{\sigma}\right)\right)^{-1/\xi}.$$

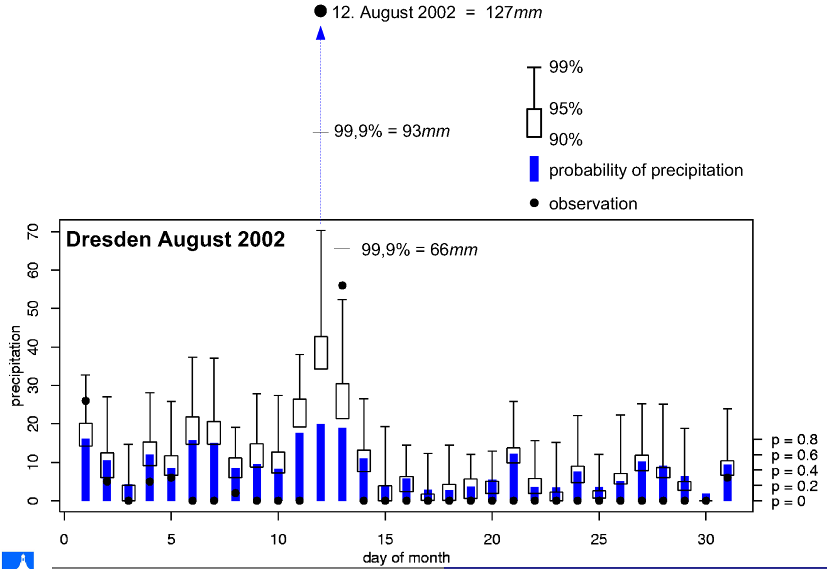
Parameters (linearly) depend on covariate $\mathbf{X}$

$\mathbf{X}$ information from (large-scale) model

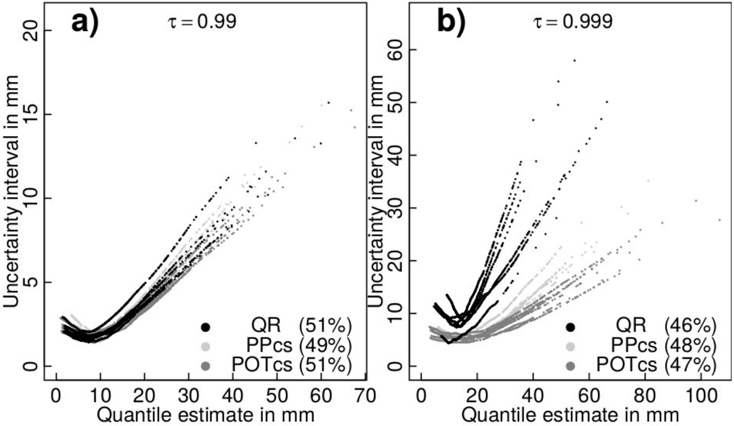
$$\mu \to \boldsymbol{\mu}^T\mathbf{X} \qquad \sigma \to \exp(\boldsymbol{\varsigma}^T\mathbf{X}) \qquad (\xi \to \boldsymbol{\xi}^T\mathbf{X})$$

Similar to ensemble model output statistics (EMOS)

Elbe Flood


12. August 2002 = 127mm
99,9% = 93mm
99%
95%
90%
probability of precipitation
observation
Dresden August 2002
99,9% = 66mm
precipitation
day of month
0
5
10
15
20
25
30
0
10
20
30
40
50
60
70
p = 0.8
p = 0.6
p = 0.4
p = 0.2
p = 0

Uncertainty of quantile estimates



Friederichs (2010)

3. Conclusions

- Proper scoring rules remain proper
  - when weighting with respect to predictive distribution as proposed in Gneiting and Ranjan (2011)
  - for conditional or censored predictive densities as proposed in Diks et al (2011)
- Postprocessing using EVT provides skilful and reliable predictive distributions for extremes
  - Reduces uncertainty in predictive distribution

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