Atmospheric Thermodynamics

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TRAINING

Contents

Preface

The current Note stems from material presented in a 1-2 hours Training course lecture on “Introduction to moist processes” that originally has been developed by Adrian Tompkins. The aim is to give a short introduction into the principles of atmospheric thermodynamics, and to present a “handy” overview and derivation of the quantities used in numerical weather prediction

The material presented is kept to a minimum, and focuses on the concept of enthalpy of moist air. This should allow the reader to elaborate on more involved thermodynamical problems.

Several textbooks dedicated to atmospheric thermodynamics exist. One might notice that some authors mainly use the notion of “adiabatic” atmospheric processes whereas others use the notion of isentropic processes throughout their derivations. Here we will use both notions.

Among the textbooks I would particularly recommend

- Dufour L. et J. v. Mieghem, 1975: *Thermodynamique de l’Atmosphère*, Institut Royal météorologique de Belgique
- Maarten Ambaum, 2010: *Thermal Physics of the Atmosphere*, John Wiley Publishers
- Sam Miller, 2015: Applied thermodynamics for Meteorologists, Cambridge Univ. Press
- Rogers and Yau, 1989: *A short course in cloud physics*, International series in natural philosophy
- Houze R., 1993: *Cloud dynamics*, Academic Press
- Emanuel K. A., 1994: *Atmospheric convection*, Oxford University Press

The book by Dufour and v. Mieghem is probably the most comprehensive and accurate textbook in atmospheric thermodynamics. Unfortunately it seems not available anymore.

1 Ideal gas law

In an ideal gas the individual molecules are considered as non-interacting. The ideal gas law stems from the law of Boyle and Mariotte saying that at constant temperature T

$$pV = cste \tag{1}$$

and the law of Gay-Lussac that describes the relation between temperature *T*, and pressure *p* for a gas with given mass m and volume *V*

$$\frac{p}{T} = cste \tag{2}$$

The ideal gas law can then be written as either

$$\begin{aligned} p\,V &= m\,R\,T \\ p\,\alpha &= R\,T; \quad \alpha = \rho^{-1} \end{aligned} \tag{3}$$

where *R* is the gas constant, ρ is density, and α the specific volume of the gas. A gas under e.g. very high pressure might not entirely follow the ideal gas law. It is then called a van der Waals gas, and for these gases an additional correction term is added to the rhs of (3).

1.1 Gas law for dry air and water vapour

Using subscripts d and v for dry air and water vapor, respectively, it follows from (3) that

$$\begin{aligned} p_d &= \rho_d\,R_d\,T \\ p_v &= \rho_v\,R_v\,T \end{aligned} \tag{4}$$

where pv is the water vapor pressure (Note that the water vapor pressure is also often denoted as e in the literature). The gas constants are given as Rd=287.06 J kg-1 K-1, and Rv=461.52 J kg-1 K-1, and one defines

$$\varepsilon = \frac{R_d}{R_v} = 0.622 \tag{5}$$

1.2 Gas law for mixture of dry air and water vapor

Dalton’s law says that when different gases are put in the same volume, then the pressure of the mixture of gases is equal to the sum of the partial pressures of the constituents. Therefore one obtains for moist air

$$p = p_d + p_v; \quad p_d = p\,N_d; \quad p_v = p\,N_v \tag{6}$$

where p is the actual atmospheric pressure, and Nd and Nv are the Mole fractions of dry air and water vapor, respectively. It follows the important relation

$$\frac{dp}{p} = \frac{dp_d}{p_d} = \frac{dp_v}{p_v} \tag{7}$$

which means that the change in water vapour pressure is proportional to the change in atmospheric pressure (e.g. when climbing to higher altitudes both the atmospheric pressure and water vapour pressure decrease). Then from (3) , (4)and (5) the gas equation for moist air is obtained as

$$p\,V = (m_d R_d + m_v R_v)\,T \tag{8}$$

2 First law of thermodynamics

The first law, or the principle of energy conservation, says that it exists a state function, the internal energy that increases according to the heat supplied and diminishes according to the work done by the system. Denoting the specific internal energy (J kg-1) as e=E/m (this is the traditional notation and e should not be confused with the water vapour pressure), the first law writes

$$de = dQ - dw = dQ - p\,d\alpha; \quad dQ <= Tds \tag{9}$$

where dQ denotes the heat supply, s the entropy, and dw the change due to the work done. Meteorologists often use the notion of heat, but formally it is better to work with entropy instead, as dQ is not a perfect differential, but ds is. The internal energy equation in its differential form can also be expressed as

$$\begin{aligned} de &= c_v dT = dQ - p\,d\alpha = Tds - p\,d\alpha \\ c_v &= \frac{\partial e}{\partial T} = \left.\frac{\delta Q}{\delta T}\right|_{\alpha}; \quad c_{vd} = \frac{5}{2}R_d \end{aligned} \tag{10}$$

where cv is the specific heat, or better heat capacity, at constant volume. The value cvd =5/2 Rd = 717.6 J kg-1 K-1 for dry air is obtained for a diatomic gas as is the atmosphere.

Next possible inequality between dQ and Tds in (9) is explained through the second law of thermodynamics.

3 Second law of thermodynamics

The second law of thermodynamics goes back to the work of Carnot, Clausius and Kelvin. It states that the entropy of the system can not diminish; it can only either remain constant or increase. For a reversible transformation one can write

$$Tds - dQ = 0 \tag{11}$$

where dQ/T is the exact differential of the state function s. It follows that for an adiabatic transformation (dQ=0), the entropy is an invariant of the system. For an *irreversible* transformation

$$Tds - dQ' = 0 \tag{12}$$

where the Clausius non-compensated heat *dQ'* has been introduced. It follows that for an irreversible adiabatic (dQ=0) transformation the entropy produced by the system is $ds = dQ'/T$.

4 Enthalpy

The beauty of the first law of thermodynamics is that all other relevant state functions can be easily derived through so called Legendre transformations. Writing

$$de = Tds - pd\alpha = Tds - d(p\alpha) + \alpha dp \qquad (13)$$

It follows that

$$d(e + p\alpha) = dh = Tds + \alpha dp \qquad (14)$$

where a new state function, the enthalpy, has been derived that has dependent variables entropy and pressure. This can also be written as

$$dh = c_p dT = Tds + \alpha dp; \quad c_p = \frac{\partial h}{\partial T} = c_v + R; \quad c_{pd} = c_{vd} + R \qquad (15)$$

with c_{pd}=7/2 R_d=1004.7 J kg^{-1} K^{-1} the heat capacity for dry air at constant pressure, as obtained from (14) and (3). The enthalpy is the preferred state function in meteorology as it uses pressure as dependent variable, and simplifies for isentropic transformations. Furthermore, the enthalpy is also the relevant function in flow processes as can be seen from the equations of motion

$$\begin{aligned} \frac{d\vec{U}}{dt} &= -\frac{1}{\rho}\vec{\nabla}p = -\alpha\vec{\nabla}p \\ \frac{d\vec{U}}{dt} &= -\vec{\nabla}h + T\vec{\nabla}s = -\vec{\nabla}h; \quad ds = 0 \end{aligned} \qquad (16)$$

Therefore, in *isentropic flow* (see Section 8.1) the acceleration of the flow is given by the gradient of the enthalpy.

5 State functions and Maxwell relations

As for the enthalpy, one can further apply Legendre transformations to change the dependent variables to derive the Helmholtz free energy *f* and the Gibbs function g_f. The Gibbs function, having dependent variables temperature and pressure, is particularly convenient to describe phase transitions. As a summary all four energy functions are listed in (17). Also, for each function a corresponding Maxwell relation is obtained stemming from the fact that the order of differentiation is irrelevant, e.g. $\partial(\partial e/\partial s)/\partial\alpha = \partial(\partial e/\partial\alpha)/\partial s$

$$\begin{aligned} & de = Tds - pd\alpha & & dh = Tds + \alpha dp \\ & \left(\frac{\partial T}{\partial \alpha}\right)_s = -\left(\frac{\partial p}{\partial s}\right)_\alpha & & \left(\frac{\partial T}{\partial p}\right)_s = \left(\frac{\partial \alpha}{\partial s}\right)_p \\ & df = -sdT - pd\alpha & & dg_f = -sdT + \alpha dp \\ & \left(\frac{\partial s}{\partial \alpha}\right)_T = \left(\frac{\partial p}{\partial T}\right)_\alpha & & \left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial \alpha}{\partial T}\right)_p \end{aligned} \qquad (17)$$

6 Humidity variables

For any intensive quantity χ (i.e. a scale invariant quantity that does not depend on the amount of substance, in contrast to an extensive quantity like mass or volume) one can write the mixture of a dry air mass and a mass of water vapour as

$$\left(m_d + m_v\right)\chi = m_v\chi_v + m_d\chi_d \tag{18}$$

in order to obtain

$$\chi = \frac{m_v}{m_d + m_v}\chi_v + \frac{m_d}{m_d + m_v}\chi_d \tag{19}$$

The specific humidity is defined as the ratio of the mass of water vapour and moist air so that we can rewrite (19) to obtain

$$\chi = q_v\ \chi_v + (1 - q_v)\ \chi_d \tag{20}$$

Note that this is also valid for temperature. However, when the air mass becomes “saturated” the mixed value has to be adjusted to take into account latent heat effects (see Sections 10, 11, 13).

The mixing ratio is defined as the ratio of water vapour and dry air. Therefore, if we divide (18) by m_d instead of m_d+m_v, one obtains the “mixed value” of the intensive quantity in terms of the mixing ratio r_v

$$\chi = \frac{1}{1 + r_v}\left(r_v\ \chi_v + \chi_d\right) \tag{21}$$

Comparing (20) and (21) one easily obtains the conversion rules from q to r and vice versa

$$q_v = \frac{r_v}{1 + r_v}; \qquad r = \frac{q_v}{1 - q_v} \tag{22}$$

There are different ways to describe the water content in the atmosphere. A list of the most common definitions is given in

Table 1.

Table 1. List of common humidity variables and their usual notations. ε=0.622 is defined in (5).

	Unit	**Definition**
Vapour pressure	Pa	e=pv
Absolute humidity	kg m-3	$\rho_v = \frac{m_v}{V}$
Specific humidity	kg kg-1	$q = q_v = \frac{m_v}{m_d + m_v} = \frac{\rho_v}{\rho} = \frac{\rho_v}{\rho_d + \rho_v} = \varepsilon\frac{e}{p - (1-\varepsilon)e} \approx \varepsilon\frac{e}{p}$
Mixing ratio	kg kg-1	$r = r_v = \frac{m_v}{m_d} = \frac{\rho_v}{\rho_d} = \varepsilon\frac{e}{p - e} \approx \varepsilon\frac{e}{p}$
Relative humidity		$RH = \frac{e}{e_s} = \frac{p_v}{p_{vs}} \approx \frac{q_v}{q_{vs}}$
Specific liquid water content	kg kg-1	$q_l = \frac{\rho_l}{\rho}$

Total water content	kg kg-1	$q_w = q_t = q_v + q_l$

7 Virtual temperature

Another way to describe the vapour content is the virtual temperature which is an artificial temperature. It describes the temperature dry air needs to have in order to have at the same pressure the same density as a sample of moist air. Definition

$$\rho = \frac{p}{R_d T_v} = \frac{p}{RT} \tag{23}$$

The probably easiest derivation of the virtual temperature is obtained by recalling that *R* of the mixture is given by (20), i.e.

$$R = q_v\ R_v + (1-q_v)R_d = R_d(1+q_v\ {}^{1-\varepsilon}\!/_{\varepsilon}) \tag{24}$$

Putting (24) into (23) and rearranging, one obtains

$$T_v \equiv T\left[1+\frac{(1-\varepsilon)}{\varepsilon}q_v\right] = T\left[1+\frac{(1-\varepsilon)r_v}{\varepsilon(1+r_v)}\right] \approx T(1+0.608q_v) \tag{25}$$

Therefore, moisture decreases the air density and increases the virtual temperature, e.g. an air parcel at *T*=300 K at q_v=5 g kg^{-1} is virtually about 1 K warmer than a reference dry air parcel.

8 Reversible adiabatic transformation without phase change

8.1 Potential temperature of dry air

The process we describe now is often called "dry" adiabatic transformation, but indeed is an isentropic transformation (*dQ* = *ds* = 0) without phase change. Considering only dry air, one obtains from the enthalpy equation (15)

$$c_{pd}dT = \alpha dp = \frac{R_d\ T}{p}dp \tag{26}$$

or

$$c_{pd}d\ln T = R_d d\ln p \tag{27}$$

which can be integrated from state $T_1 = T$, $p_1 = p$ to a state $T_0 = \theta$, p_0 = 1000 hPa to obtain

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa};\quad \kappa = \frac{R_d}{c_{pd}} = \frac{2}{7} \tag{28}$$

Where θ, a quantity that usually increases with height, is referred to as the potential temperature and is also plotted as such in thermodynamic diagrams (see below). Finally, differentiating (28) and taking into account (15), one obtains the important relation between entropy and the potential temperature

$$ds = c_{pd}d\ln\theta \tag{29}$$

which states that *ds* = 0 in isentropic flow ($d\theta$ = 0).

8.2 Poisson relation and speed of sound

With the aid of (28) , using $T_0 = \theta$ and (3) one can also verify that

$$\left(\frac{p}{p_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{C_{pd}/C_{vd}} \tag{30}$$

which is the Poisson relation that allows to compute the speed of sound *c* in dry air as

$$c^2 = \frac{\partial p}{\partial \rho} = \frac{c_{pd}}{c_{vd}} R_d T \tag{31}$$

that for *T* = 300 K is *c* = 347.2 m s^{-1} or roughly 331 m s^{-1} for *T* = 273.16 K=0°C.

8.3 Potential temperature of moist air

Considering a system that does not change mass and heat with its environment, then

$$dH - Vdp = 0 \tag{32}$$

where *H* is the enthalpy of dry air and water vapour. Taking into account (8) and (26) this can be written as

$$(m_d c_{pd} + m_v c_{pv})dT = (m_d R_d + m_v R_v)\, T\, \frac{dp}{p} \tag{33}$$

Where c_{pv} = 1846.1 J kg $^{-1}$ K^{-1} is the heat capacity of water vapour. Dividing by md or md+mv gives

$$\begin{gathered}(c_{pd} + r_v c_{pv})dT = (R_d + r_v R_v)\, T\, \frac{dp}{p} \\ [q_v c_{pd} + (1-q_v)c_{pv}]dT = [q_v R_d + (1-q_v)R_v]\, T\, \frac{dp}{p}\end{gathered} \tag{34}$$

from which it follows that

$$\theta_m = T\left(\frac{p_0}{p}\right)^{\kappa}; \quad \kappa = \frac{R_d + r_v R_v}{c_{pd} + r_v c_{pv}} = \frac{R}{c_p} \tag{35}$$

Note that we have chosen the subscript m as the subscript v is normally reserved for the virtual potential temperature where the density effect of vapor is generally only considered through the virtual temperature effect but not through the effect on κ

$$\theta_v = T_v\left(\frac{p}{p_0}\right)^{\kappa} \tag{36}$$

8.4 Dry static energy

Instead of the potential temperature of dry air one often uses the dry static energy Replacing dp in (26) by the hydrostatic approximation $dp = -g\rho dz$, with $g = 9.81 \text{ m s}^{-2}$, the gravity acceleration, one obtains

$$c_{pd} dT + g dz = ds_z = 0 \tag{37}$$

It follows that in an atmosphere in hydrostatic equilibrium the dry static energy $s_z = c_p T + gz$ (often it is denoted by s, but this notation has been reserved for the entropy) is conserved during adiabatic ascent without phase change. Furthermore, in contrast to θ it is linear in T and z, and therefore a preferred quantity for numerical weather prediction, in particular convection parametrization.

Finally, with the aid of (37) we can also derive the dry adiabatic lapse rate of the atmosphere

$$\frac{dT}{dz} = -\frac{g}{c_{pd}} \tag{38}$$

which amounts to a temperature decrease of 0.98 K per 100 m.

9 Reversible adiabatic transformation with phase change

9.1 Specific heat of phase change

Before presenting the differential enthalpy equation including phase changes, we have to introduce the latent heats. These correspond to the heat, or additional enthalpy required, to pass from state l, e.g. liquid water having enthalpy h_l to a state v, e.g water vapour having enthalpy h_v

$$L_v = h_v - h_l = h_v(T_{00}) + c_{pv}(T - T_{00}) - h_l(T_{00}) - c_l(T - T_{00}) \tag{39}$$

Following this procedure one can readily derive the expressions for the latent heat of vaporization L_v, the latent heat of sublimation (ice to vapour) L_s, and the latent heat of melting Lm

$$\begin{aligned} L_v &= L_v(T_{00}) + (c_{pv} - c_l)(T - T_{00}) \\ L_s &= L_s(T_{00}) + (c_{pv} - c_i)(T - T_{00}) \\ L_m &= L_s - L_v \\ T_{00} &= 273.16\,K; \quad L_v(T_{00}) = 2.5008 \times 10^6\ J\ kg^{-1}; \quad L_s(T_{00}) = 2.8345 \times 10^6\ J\ kg^{-1} \end{aligned} \tag{40}$$

where T_{00} is the triple point temperature where all three phases coexist, and where the heat capacities of liquid water and ice are given as $c_l = 4218 \text{ J kg}^{-1} \text{ K}^{-1}$, and $c_i = 2106 \text{ J kg}^{-1} \text{ K}^{-1}$.

9.2 Liquid phase

Equivalent to (32) the differential enthalpy equation for a closed system containing dry air, water vapour and liquid water can be written as

$$dH - Vdp = 0 \tag{41}$$

from which it follows using (33) and (39) that

$$
\begin{aligned}
&(m_d c_{pd} + m_v c_{pv} + m_l c_l)dT = (m_d R_d + m_v R_v)\,T\,\frac{dp}{p} - L_v dm_v; \quad or \\
&(m_d c_{pd} + m_v c_{pv} + m_l c_l)dT = (m_d R_d + m_v R_v)\,T\,\frac{dp}{p} + L_v dm_l
\end{aligned}
\tag{42}
$$

Note that for the condensation process $dm_v < 0$ and $dm_l > 0$. Using (40), (42) can also be written as

$$
(m_d c_{pd} + m_w c_l)dT = (m_d R_d + m_v R_v)\,T\,\frac{dp}{p} - d(L_v m_v); \quad m_w = m_v + m_l \tag{43}
$$

where m_w is the total water mass. Dividing by md one obtains

$$
c_p dT = (R_d + r_v R_v)\,T\,\frac{dp}{p} - d(L_v r_v); \quad c_p = c_{pd} + r_w c_l \tag{44}
$$

where c_p is the heat capacity of moist air containing liquid water.

One can also divide (42) by $m_d + m_v$ to obtain

$$
c_p dT = [(1-q_v)R_d + q_v R_v]\,T\,\frac{dp}{p} - d(L_v q_v); \quad c_p = (1-q_v)c_{pd} + q_w c_l \tag{45}
$$

The notation dr_{vs} instead of dr_v, and dq_{vs} instead of dq_v, is actually more appropriate as the phase change from vapor to liquid only occurs when the mixing ratio exceeds the saturation value (see Section 10).

9.3 Equivalent potential temperature and moist static energy

The equivalent potential temperature is readily obtained by integration of (44). It is the temperature obtained by a parcel during a fictive adiabatic process where all the water vapour it contains has been condensed

$$
\theta_e = T\left(\frac{p_0}{p}\right)^{\kappa} \exp(L_v r_v / c_p T); \quad \kappa = \frac{R_d}{c_p}; c_p = c_{pd} + r_w c_l \tag{46}
$$

Often c_p is replaced here by its constant dry value as it is the case for a pseudo-adiabat (a non reversible process, see Section 11.3). However, even if for T in the exponential very often the actual environmental temperature is used, strictly one should use the temperature at the isentropic condensation level (see Section 11.3). For a discussion of various definitions of the equivalent potential temperature the reader is referred to Betts (1973).

The moist static energy hz is derived from (44) using (24), repeating the procedure as for the dry static energy (Section 8.4)

$$
\begin{aligned}
&h_z = c_p T + (1+r_v)gz + L_v r_v \approx c_{pd} T + gz + L_v r_v; \quad c_p = c_{pd} + r_w c_l \\
&h_z = c_p T + gz + L_v q_v \approx c_{pd} T + gz + L_v q_v; \quad c_p = (1-q_v)c_{pd} + q_w c_l
\end{aligned}
\tag{47}
$$

It is a conserved quantity in a reversible adiabatic process, and due to its linearity, a widely used quantity in convection computations.

9.4 Liquid and ice phase

In analogy with the preceding Section one can extend (42) for the ice phase to obtain

$$\begin{aligned}
&(m_d c_{pd} + m_v c_{pv} + m_l c_l + m_i c_i)dT = (m_d R_d + m_v R_v)\,T\,\frac{dp}{p} - L_v dm_v + L_m dm_i; \quad or \\
&(m_d c_{pd} + m_v c_{pv} + m_l c_l + m_i c_i)dT = (m_d R_d + m_v R_v)\,T\,\frac{dp}{p} + L_v dm_l + L_s dm_i;
\end{aligned} \tag{48}$$

where m_i is the mass of ice. Then the conserved quantities ice-liquid potential temperature and a “liquid-water static energy” can be derived. We only give here the expression for the “liquid-water static energy” as

$$h_{zil} = c_p T + (1 + r_v) gz - L_v r_l - L_s r_i \approx c_{pd} T + gz - L_v r_l - L_s r_i; \quad c_p = c_{pd} + r_w c_l \tag{49}$$

For further accurate formulations of the equivalent potential temperature and enthalpy including the ice phase the reader is refered to Bolton (1980) and Pointin (1984).

10 Clausius Clapeyron equation

Consider a closed system where the liquid and gas phase of a substance, e.g. water, are in equilibrium, i.e. as many molecules leave the fluid to go into the vapour as vice versa. Then for this state the specific Gibbs functions g_v and g_l must be equal. From (17) it then follows that

$$\frac{dp_v}{dT} = \frac{s_v - s_l}{\alpha_v - \alpha_l} \tag{50}$$

Furthermore, using

$$s_v - s_l = \frac{h_v - h_l}{T} = \frac{L_v}{T} \tag{51}$$

one obtains the Clausius-Clapeyron equation

$$\frac{dp_v}{dT} = \frac{L_v}{T(\alpha_v - \alpha_l)} \approx \frac{L_v p_v}{R_v T^2} \tag{52}$$

where the specific volume of water has been neglected. It is important that there is no mention of air, only water substance in the derivation. Therefore, the commonly perceived fact that “air holds water” is wrong, the air does not hold water!

The problem in integrating the Clausius-Clapeyron equation lies in the temperature dependence of L_v. Assuming a constant value (52) can be readily integrated to obtain

$$\ln\left(\frac{p_{vs}}{p_{vs0}}\right) = \frac{L_v}{R_v}\left(\frac{1}{T_{00}} - \frac{1}{T}\right) \tag{53}$$

where the integration constant p_{vs0} is evaluated for the triple point temperature T_{00} = 273.16 K as p_{vs0} = 6.112 hPa. Empirical accurate integration formula of (52) for the water vapor saturation pressure over liquid water and ice have been computed by Thetens

$$p_{vs} = p_{vs0} \exp^{17.502\,(T-T_{00})/(T-32.19)}; \quad \text{liquid water}$$
$$p_{vsi} = p_{vs0} \exp^{22.587\,(T-T_{00})/(T+0.7)}; \quad \text{ice} \qquad (54)$$

In atmospheric models one typical interpolates between the saturation pressure over water and ice for temperatures below T_{00} to account for the fact that all 3 phases can be present. The water vapour saturation curve as a function of temperature is plotted in Figure 1. Two interesting final remarks can be made

- Due to the non-linearity of the saturation curve, a mixture of two unsaturated parcels can be oversaturated, as illustrated in Figure 1.
- The *boiling temperature* of water is the temperature where the pressure of the gas bubbles equals the atmospheric pressure. Therefore, from inversion of (54) we get for an atmospheric pressure of 1013 hPa a boiling temperature of 99.5 °C – everybody knows it should be something like 100 °C, but not why!

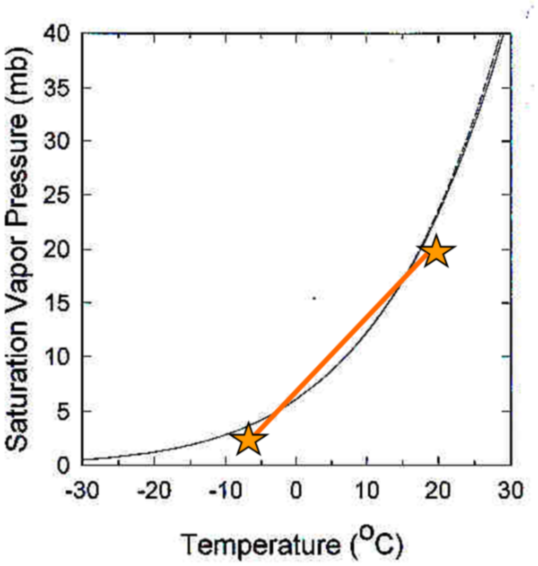

Figure 1. Water vapor saturation pressure as a function of temperature (°C). Also plotted are two unsaturated parcels (stars). As their mixture is laying on a straight line, it is shown that due to the non-linearity of the saturation curve a mixture of two unsaturated parcels can be oversaturated.

11 Ways of reaching saturation

An air parcel can reach saturation through

- Diabatic (external) cooling, e.g. by radiation
- Addition of moisture through evaporation of falling precipitation, turbulent mixing, differential advection, or surface moisture fluxes
- Cooling during isentropic ascent

The processes under the first two items are supposed to be isobaric processes (no change in pressure), whereas isentropic lifting implies the air parcel undergoes a change in height (pressure).

11.1 Dew point temperature

The dew point temperature is the temperature to which a parcel must be cooled (e.g. by radiation) in order to be saturated, i.e. its specific humidity or mixing ratio must equal the saturation specific humidity (mixing ratio)

$$q_v = q_{vs}(T_d) \tag{55}$$

One can solve this equation for T_d by inverting the saturation formula (54).

11.2 Wet bulb temperature

The wet bulb temperature T_w is the temperature to which air may be cooled at constant pressure by evaporation of water into it until saturation is reached. It can be solved for graphically (see Section 12.2) or numerically be solving e.g. the simplified moist enthalpy equation

$$c_p dT = -L_v dq_{vs} = -L_v(q_{vs}(T) - q_v) \tag{56}$$

This equation is non-linear and therefore either requires an iterative procedure or a linearized formulation (for the latter see Section 13).

11.3 Isentropic (or adiabatic) condensation temperature

As (unsaturated) moist air expands (e.g. through vertical motion), it cools adiabatically conserving θ. Eventually saturation pressure is reached. The temperature and pressure at that level $T = T_{c,p} = p_c$ are known as the “isentropic condensation temperature” and “pressure”, respectively. The level is also known as the “Lifting Condensation Level”. If expansion continues, condensation will occur (assuming that liquid water condenses efficiently and no super saturation can persist), thus the temperature will decrease at a slower rate, the moist adiabatic lapse rate. The isentropic condensation temperature can be easily determined graphically (see Section 12.1) or numerically, by searching, starting from the surface (pressure p, temperature T), for the layer where first $q_v(p) >= q_{vs}(p_c)$. T_c is then obtained as

$$T_c = \theta(T, p)\left(\frac{p_c}{p_0}\right)^{\kappa} \tag{57}$$

Formula also exist for an accurate direct numerical computation (e.g. Davies-Jones, 1983) of (T_c,p_c) for given departure properties (T,p)

11.4 Pseudo-adiabatic processes

When lifting a parcel one has to make a decision concerning the condensed water. Does it falls out instantly or does it remain in the parcel? If it remains, it will have an important effect on parcel buoyancy (see Lecture Note on convection), and also the heat capacity of liquid water needs to be accounted for. Furthermore, once the freezing point is reached, ice processes would need to be taken into account. These are issues concerning microphysics, and dynamics. The air parcel history will depend on the situation. Therefore, often one takes a simple approach known as the “pseudo adiabatic process”, a non-reversible process, where one assumes that all condensate formed immediately leaves the parcel. The pseudo-adiabatic, approximation can be qualified as a “good” approximation, it is in any case a much better approximation than assuming a reversible adiabat, i.e all condensates remains in the parcel. Furthermore, a thermodynamic diagram, a 2D diagram (see below) cannot hold a third dimension (condensate), and therefore always pseudoadiabats are plotted.

12 Energy diagrams – Tephigram

The Tephigram is a meteorological energy diagram with an orthogonal coordinate system consisting of temperature T (°C) and potential temperature θ (°C) (dry adiabat). It is illustrated in Figure 2. Knowing T and θ one can compute pressure p. The isobars appear as quasi-horizontal lines in Figure 2. Furthermore, knowing T and p one can compute the saturation specific humidity q_{vs} or mixing ratio r_{vs}, and the moist adiabats or more precisely pseudoadiabats.

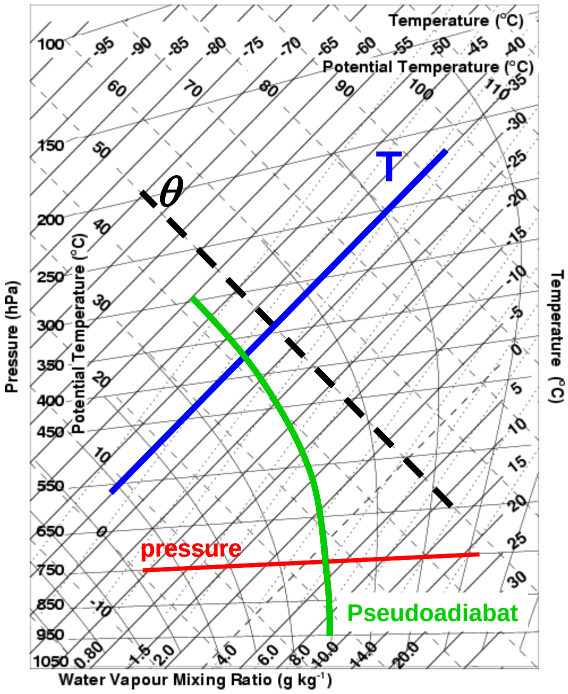

Figure 2. Tephigram with highlighted isotherm (blue), isentrope (iso-theta) (dashed-black), isobar (red), and saturation specific humidity (dotted black).

Another property of the Tephigram becomes apparent when writing the enthalpy equation (15) as

$$\oint dQ = \oint c_{pd} T \left(\frac{dT}{T} - \frac{R_d}{c_{pd}} \frac{dp}{p} \right) = c_{pd} \oint T d \ln \theta \qquad (58)$$

from which it follows that in this particular coordinate system areas T dlnθ are areas of equal energy (heat content).

12.1 Determine isentropic condensation temperature

During dry adiabatic ascent, both θ and the specific humidity of the parcel are conserved. Therefore, the isentropic condensation temperature T_c is graphically obtained in Figure 3 as the intersection of the dry adiabat through the departure point (T,p) with the saturation specific humidity line through p having the value $q_{sv}=q_v(p)$ or in other words having the coordinates (T_d,p). Then at the level p_c, the lifting condensation level, the parcel is actually saturated, so that its humidity $q_v(p)$ is equal to the saturation value, and its temperature T_c and dewpoint temperature become equivalent.

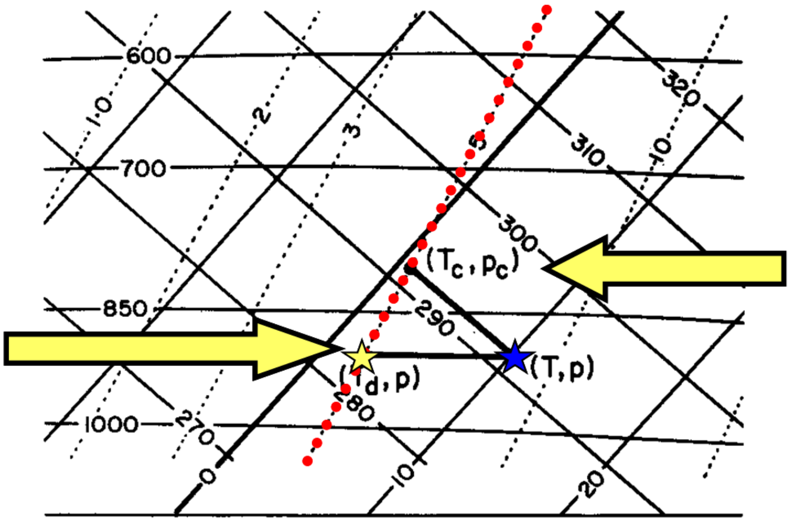

Figure 3. Graphical determination of the isobaric condensation temperature T_c: It is obtained as the intersection of the dry adiabat through the departure point (T,p) with the saturation specific humidity line (dotted red) through the point (q_v,p) which is equivalent to the dewpoint (T_d,p). Note that a sounding is either given as (T(p),T_d(p)) or (T(p),q_v(p)).

12.2 Wet-bulb temperature

The wet-bulb temperature T_{sw} is the temperature of an air parcel brought to saturation by e.g. evaporation of rain. It therefore plays an important role in the computation of convective downdraughts driven by cooling through evaporating precipitation. The graphical procedure to determine T_{sw} is illustrated in Figure 4. One determines first, as in Figure 3, the lifting condensation level, and then follows the moist adiabat through the lifting condensation level down to the departure level p.The wet-bulb potential temperature θ_w is obtained by further extending the moist-adiabat to the 1000 hPa pressure level.

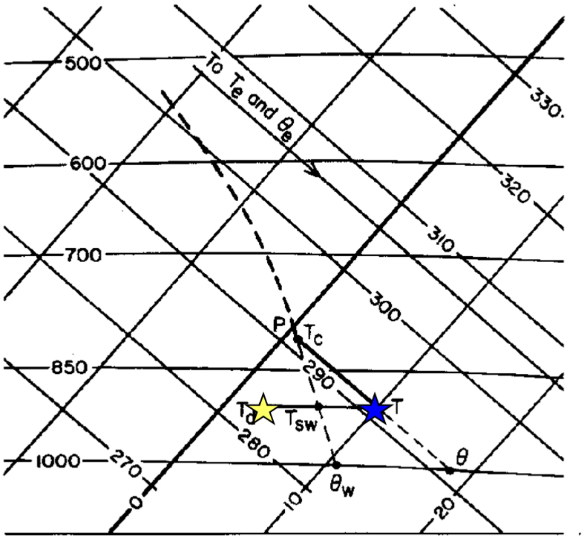

Figure 4. Determination of the wet-bulb temperature T_{sw}. As in Figure 3, one determines first the lifting condensation level (T_c,p), but then extends the moist adiabat through the lifting condensation level down to the departure level p, where the temperature T_{sw} can be read from the diagram.

12.3 Equivalent potential temperature

The equivalent potential temperature is conserved during moist adiabatic ascent, and also in the presence of precipitation. This makes it an attractive variable for moist convective modelling, including saturated updraughts and downdraughts. Its graphical determination is illustrated in Figure 5 and can be summarised as follows. One follows the moist adiabat through the lifting condensation level upward until all water vapour has been removed (condensed), i.e. the slope of the moist adiabat becomes equal to that of a dry adiabat. Then one descends along the dry adiabat down to the departure level to obtain T_e or further to the 1000 hPa level to obtain the equivalent potential temperature θ_e.

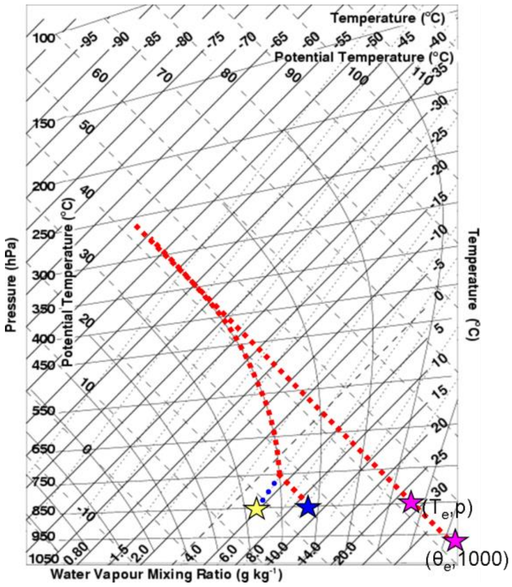

Figure 5. Graphical determination of the equivalent temperature T_e and equivalent potential temperature θ_e. One follows the moist adiabat through the lifting condensation level upward until all water vapour has been condensed, and then descends along a dry adiabat to the departure level or the 1000 hPa level, respectively.

13 Saturation adjustment – numerical procedure

Last not least we still need an efficient numerical procedure to solve the saturation adjustment problem. The procedure uses a simplified form of the moist enthalpy equation (42) and (45) at constant pressure (dp = 0). The initial temperature T and specific humidity q_v are known

Given T, q_v

check if $q_v > q_{vs}(T)$ then

solve for adjusted T^*, q_v^*

so that $q_v^* = q_{vs}(T^*)$

$q_l = q_v - q_{vs}(T^*)$

Using $c_p dT = -L_v dq_{vs}$

The last relation can be linearised with the aid of a first order Taylor expansion around the initial temperature T to yield

$$c_p(T^* - T) = L_v\left[q_v - \left\{q_{vs}(T) + \left.\frac{dq_{vs}}{dT}\right|_T (T^* - T) + O(2)\right\}\right]$$

$$T^* = T + \frac{L_v}{c_p}\frac{q_v - q_{vs}(T)}{1 + \frac{L_v}{c_p}\left.\frac{dq_{vs}}{dT}\right|_T} \qquad (59)$$

(59) can be solved using (52) and the definition of q_{vs} in

Table 1. Once T^* is known one can compute q_v^* and q_l. This computation is very accurate. However, for accuracy of order (10^{-2} K) it can be done twice. Note that in (59) we have used a general c_p, so that moist effects could possibly be included. The formula can also be extended for the ice phase by using "mixed" or interpolated values for L_v and q_{sv}, e.g. as a function of temperature.

Finally, note that the denominator in (59) has value between 1 and 2, so that the actual temperature increment T^*-T has value $0 < T^* - T < \frac{L_v}{c_p}(q_v - q_{vs}(T))$ …… loosely speaking it is somewhere in the middle because of the latent heat release during condensation, changing in turn the saturation value. This fact is graphically illustrated in Figure 6.

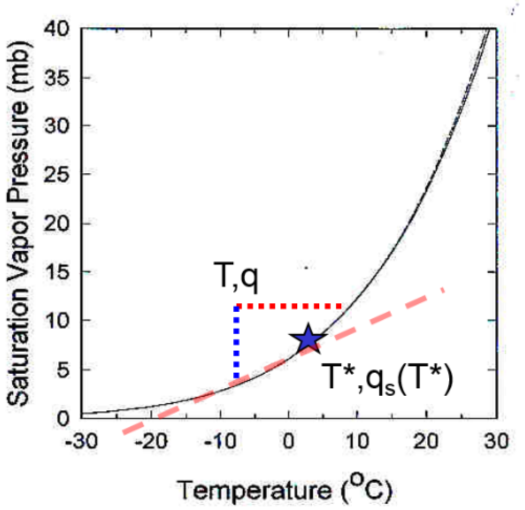

Figure 6. Illustration of the adjustment procedure from an initial oversaturated state (T,q_v) to an adjusted state (T*,q_{vs}(T*)). The result T* is obtained through a linearization of the saturation curve at T.

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List of Symbols

c	Speed of sound	m s^{-1}
c_p	Heat capacity (specific heat) at constant pressure	J kg^{-1} K^{-1}
c_{pd}	Heat capacity of dry air	
c_{pv}	Heat capacity of water vapour	
c_l	Heat capacity of liquid water	
c_i	Heat capacity of ice	
c_p	Heat capacity (specific heat) at constant volume	J kg^{-1} K^{-1}
E	Internal energy	J
e	Internal energy, specific	J kg^{-1}
f	Helmholtz free energy, specific	
g_f	Gibbs function, specific	
h	Enthalpy, specific	
h_z	Moist static energy, specific	
h_{zil}	Liquid water static energy, specific	
H	Enthalpy	J
g	Gravity constant	m s^{-2}
L_m	Latent heat of melting	J kg^{-1}
L_s	Latent heat of sublimation	
L_v	Latent heat of vaporisation	
m	Mass	kg
m_d	Mass of dry air	
m_i	Mass of ice	
m_l	Mass of liquid water	
m_v	Mass of water vapor	
m_w	Total mass of water (vapour+liquid)	
N_d	Mole fraction of dry air	unitless
N_v	Mole fraction of water vapor	
p	Pressure (total pressure of atmosphere)	Pa
p_d	Pressure of dry air	
p_v,e	Pressure of water vapour	
p_{vs}	Saturation water vapour pressure	
q_v	Specific humidity	unitless
q_{vs}	Saturation specific humidity	
ql	Specific humidity of liquid water	
q_w	Total water specific humidity	
R	Gas constant	J kg^{-1} K^{-1}
R_d	Gas constant for dry air	
R_v	Gas constant for water vapour	
RH	Relative humidity	unitless
r_i	Ice mixing ratio	unitless
r_v	Water vapour mixing ratio	
r_w	Total water mixing ratio	
s	Entropy, specific	J kg^{-1} K^{-1}
s_z	Dry static energy, specific	J kg^{-1}

$\boldsymbol{T}$	Temperature	K
$\boldsymbol{T_c}$	Adiabatic condensation temperature	
$\boldsymbol{T_d}$	Dewpoint temperature	
$\boldsymbol{T_e}$	Equivalent temperature	
$\boldsymbol{T_w}$	Wet bulb temperature	
$\boldsymbol{Tv}$	Virtual temperature	
$\boldsymbol{V}$	Volume	m^3
$\boldsymbol{z}$	Height	m
$\boldsymbol{\alpha}$	Specific volume	$m^3\ kg^{-1}$
$\boldsymbol{\varepsilon}$	R_d/R_v	unitless
$\boldsymbol{\kappa}$	R/c_p	
$\boldsymbol{\rho}$	Density	$kg\ m^{-3}$
$\boldsymbol{\rho_d}$	Density of dry air	
$\boldsymbol{\rho_v}$	Density of water vapour	
$\boldsymbol{\rho_l}$	Density of liquid water	
$\boldsymbol{\theta}$	Dry Potential temperature	K
$\boldsymbol{\theta_m}$	Potential temperature of moist air	
$\boldsymbol{\theta_e}$	Equivalent Potential temperature	
$\boldsymbol{\theta_v}$	Virtual Potential temperature	
$\boldsymbol{\theta_w}$	Wet bulb quivalent Potential temperature	