Horizontally-explicit vertically-implicit time-stepping methods for NWP and climate models

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ABSTRACT

It is anticipated that computational methods that require local, and not global, data will be best suited for exploiting the distributed-memory of future massively parallel computers. Consequently, for atmospheric models, which retain whole grid-columns on each processor, horizontally-explicit vertically-implicit (HEVI) time-stepping schemes warrant further attention. This discussion provides a short review of the "split-explicit" time-stepping method, which is the most widespread HEVI approach in current atmospheric models; and follows with a summary of some recent analyses of a simpler HEVI approach with avoids the split-explicit sub-stepping. The analyses identify a number of Runge-Kutta implicit-explicit schemes for HEVI solution of the atmospheric system, and consider their performance in terms of stability, accuracy and cost.

The analyses identify three Runge-Kutta schemes that offer good stability and accuracy; and, therefore, good relative efficiency (defined as accuracy at a given cost). All three schemes have been developed for use in atmospheric models. By contrast, schemes identified from the wider literature perform less well: higher accuracy but for impractically high cost; or an inability to guarantee stability for all wavenumbers within some restricted time-step.

1 Background

Consider the atmospheric system to be described most simply by

$$\frac{\partial F}{\partial t} = g\left(F\right), \tag{1}$$

where F represents the N normal modes of the system, e.g. $[\mathbf{u}, T, p]^{\mathrm{T}}$, and g represents the associated forcing terms, e.g. rotation, advection, gravity, pressure-gradient forces, which define F's evolution in time, t.

For a discretised numerical model, the time-stepping scheme provides a method for approximating (1) by using model solutions at discrete times $j\Delta t$ for integer j. If the latest model solution describes the state of the atmosphere at time $n\Delta t$, then a time-stepping scheme yields the solution at the next model time-step by approximating (1) by

$$\frac{\phi^{n+1} - \phi^{\star}}{\alpha\Delta t} = \tilde{g}\left(\overline{\phi}\right),$$

where $\phi^{\star}$ represents some combination of known model solutions, ϕ^{j} for $j \leq n$ and $\alpha\Delta t$ indicates the time-interval over which the approximation is made. On the right-hand side, $\overline{\phi}$ also represents some combination of model solutions, but which can include the upcoming solution, i.e. ϕ^{j} where $j \leq (n+1)$, and $\tilde{g}$ indicates the discretised forcing terms.

An *explicit* time-stepping scheme is defined as one for which the right-hand side forcing is evaluated using only model solutions from previous time-steps, i.e. $\overline{\phi} = \overline{\phi}\left(\phi^{n}, \phi^{n-1}, \phi^{n-2}, \ldots\right)$. Explicit schemes yield a system of equations that is simple to construct and can be solved directly, but suffer severe stability constraints that limit the length of time-step.

By contrast, *implicit* schemes are those which include the solution at the new time-level in the estimate of the right-hand side, i.e. $\overline{\phi} = \overline{\phi}\left(\phi^{n+1}, \phi^{n}, \phi^{n-1}, \ldots\right)$. Implicit schemes provide stability unlimited by the length of time-step, but the resulting system of equations is more difficult to construct and can require an iterative approach to solve.

As well as stability, the extent to which a time-stepping method (be it explicit or implicit) can *accurately* estimate the new model solution on a given mesh also depends on the length of time-step. Specifically, the accuracy and stability of any method can be determined by the Courant numbers

$$c\frac{\Delta t}{\Delta X}$$

associated with the solutions to the model problem, where c is the speed of propagation and ΔX indicates the grid-spacing. In general terms, as the Courant number grows, the time-stepping method yields poorer accuracy and, for explicit methods, becomes unstable. For a thorough introduction to time-stepping schemes for atmospheric models, see e.g. Durran (2010).

Returning to (1), the atmospheric system is in fact better described by

$$\frac{\partial F}{\partial t} = f(F) + s(F), \tag{2}$$

where $||f|| \gg ||s||$ (by some norm). The difference in magnitude of f and s comes from two aspects:

1. solutions to the continuous model (2) comprise fast and slow modes, i.e. f and s can describe processes that differ by orders of magnitude with respect to the timescale of their propagation; and in addition,

2. in the discretised model, the grid-spacings used to resolve the horizontal (x) and vertical (z) directions are highly anisotropic, such that $\Delta x \gg \Delta z$. For a typical global NWP model, $\Delta x = \mathrm{O}(10)$ km, while $\Delta z = \mathrm{O}(10\,\mathrm{m})$ near the Earth's surface. Since the terms on the right-hand side of (2) include spatial gradients, then if f represents contributions from the vertical direction and s from the horizontal, the mesh-anisotropy leads to a separation of scales.

Suppose (2) describes the nonhydrostatic equations that govern many current-day global NWP and climate models. Dispersion relation analyses of such a system identify the major dry dynamical processes to be (from slowest to fastest):

- rotation
- advection (with speed U)
- gravity waves
- acoustic waves (with speed c_s)

where $c_s \gg U$.

So, from item 1., we see that in a given direction (e.g. horizontal) the Courant numbers associated with acoustic waves and advection are very different:

$$c_s\frac{\Delta t}{\Delta x} \gg U\frac{\Delta t}{\Delta x}; \tag{3}$$

and from item 2., for any given process, the Courant numbers associated with the direction of propagation are very different:

$$c_z \frac{\Delta t}{\Delta z} \gg c_x \frac{\Delta t}{\Delta x}. \tag{4}$$

Since the accuracy and stability of a time-stepping method are determined by its ability to handle the largest Courant numbers associated with the choice of time-step and grid-spacing, then we must consider how to tackle the difference in scales from both (3) and (4).

First, consider the physical implications of (3) and (4). It is commonly believed that acoustic waves have little direct meteorological significance. We can therefore aim to choose a numerical approach that exploits this lack of importance — either by removing the acoustic waves altogether or choosing to represent them with a scheme that provides a cheap but inaccurate approach. In contrast, the mesh anisotropy that yields (4) is a direct result of the need for much finer grid-spacings to sufficiently accurately resolve the larger gradients in the vertical than in the horizontal. The chosen numerical approach must therefore offer both a stable and accurate method for solving the physically important processes that yield the largest Courant numbers in (4).

In brief, current atmospheric models address these problems in a number of ways:

- tackling (3) by choosing a system of filtered (sound-proof) governing equations (see e.g. Klein *et al.*, 2014; Smolarkiewicz *et al.*, 2014, in these Proceedings);
- tackling both (3) and (4) with "semi-implicit semi-Lagrangian" solutions (e.g. Robert *et al.*, 1985), whereby an implicit scheme is used to solve the fastest but least physically significant modes (commonly, acoustic and gravity terms, Cullen, 1990) avoiding the tightest stability constraints, and the semi-Lagrangian method stably handles the large Courant numbers associated with the physically important process of advection. The combined scheme enables long time-steps that yield sufficiently short wall-clock times for operational forecast deadlines. The semi-implicit approach yields a 3D elliptic equation for the pressure increment that must be solved at every time-step. The problem describes a global coupling over the domain, and its solution necessarily requires non-local data, which poses an increasingly difficult problem as computer memory becomes more distributed;
- choosing a "HEVI" (horizontally-explicit vertically-implicit) approach (e.g. Ikawa, 1988) to tackle (4), and accepting the stablity constraint that comes from the fastest moving horizontal waves in (3).

2 HEVI approaches

2.1 Split-explicit approaches

The widely used "split-explicit" (or "time-splitting") approach (proposed in Klemp and Wilhelmson, 1978) is an example of a HEVI method. Vertically-propagating fast waves are solved with a simple (typically, trapezoidal, see e.g. Durran, 2010) implicit scheme, yielding a tridiagonal system of equations for each vertical column, which can be efficiently solved with no cross-processor communication (since for atmospheric models, domain decomposition is typically limited to the horizontal, preserving entire columns on each processor). For the horizontal, the fast acoustic waves are solved on an appropriately short sub-step ($\Delta\tau$) within a longer time-step ($\Delta t = M\Delta\tau$) that resolves the slower, more physically important terms.

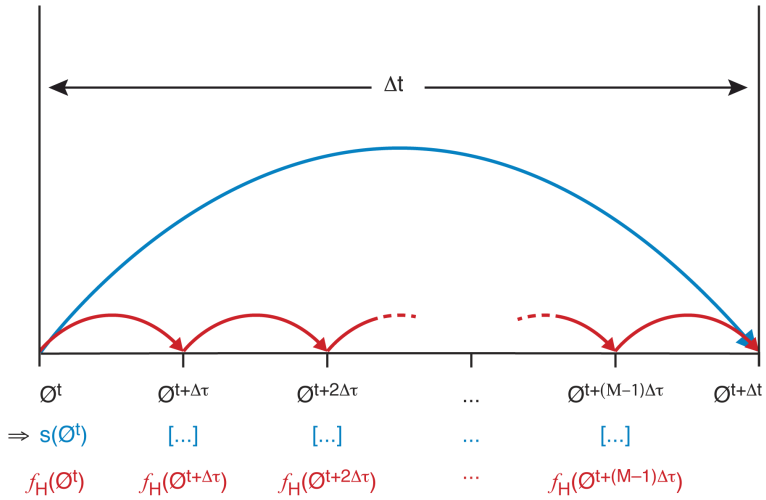

Figure 1: An example split-explicit approach for solving the horizontal terms in (5): at time t, terms s and f_H can be computed using model solution ϕ^t to give a new solution at time $t+\Delta\tau$. At this new model time, the contribution from f_H is updated using $\phi^{t+\Delta\tau}$ and is combined with the previously computed $s(\phi^t)$ to generate the solution at time $t+2\Delta\tau$. The f_H contributions continue to be updated at each sub-step and combined with the fixed contribution from s until time $t+M\Delta\tau = t+\Delta t$.

The system solved by the split-explicit approach extends from (2) to

$$\frac{\partial F}{\partial t} = f_V(F) + f_H(F) + s(F), \tag{5}$$

with fast contributions acting in the vertical (f_V) and horizontal (f_H). A simple example of the method of time-splitting that treats the solutions of the horizontal terms (f_H and s) is described pictorally in Figure 1. The approach achieves efficiency by only solving the s terms once per model time-step, Δt. Contributions from s include advection and diffusion terms that require a relatively large stencil of data and are therefore more computationally expensive. Meanwhile, the f_H terms are updated every sub-step, $\Delta\tau$, but involve only immediate neighbour data-points to compute local gradients.

Typically, all split-explicit models make the same choice of schemes for resolving the fast components (see e.g. Klemp and Wilhelmson, 1978): the trapezoidal scheme with small off-centering to enhance stability for f_V and the forward-backward scheme (Mesinger, 1977) for terms in f_H. For s, a scheme that offers good accuracy and an acceptable window of stability (in terms of time-step length) is required. The example illustration in Figure 1 implies a time-stepping scheme that is only first-order accurate for solving s. In fact, split-explicit models tend to use either leapfrog (as was proposed originally in Klemp and Wilhelmson, 1978) or a 3-stage Runge-Kutta scheme (Wicker and Skamarock, 2002). Simple illustrations of the use of these schemes in a split-explicit model are presented in Figure 2.

The split-explicit approach is employed in a number of current models — both established, e.g. WRF (Klemp *et al.*, 2007), COSMO (Baldauf, 2008, 2010), JMA-NHM (Saito *et al.*, 2006) models; and "next-generation" models: MPAS (Skamarock *et al.*, 2012, 2014, in these proceedings), NICAM (Tomita and Satoh, 2004).

Since all the computations are local (regardless of choice of time-stepping scheme), the split-explicit approach offers good scalability on massively parallel machines (Michalakes *et al.*, 2008).

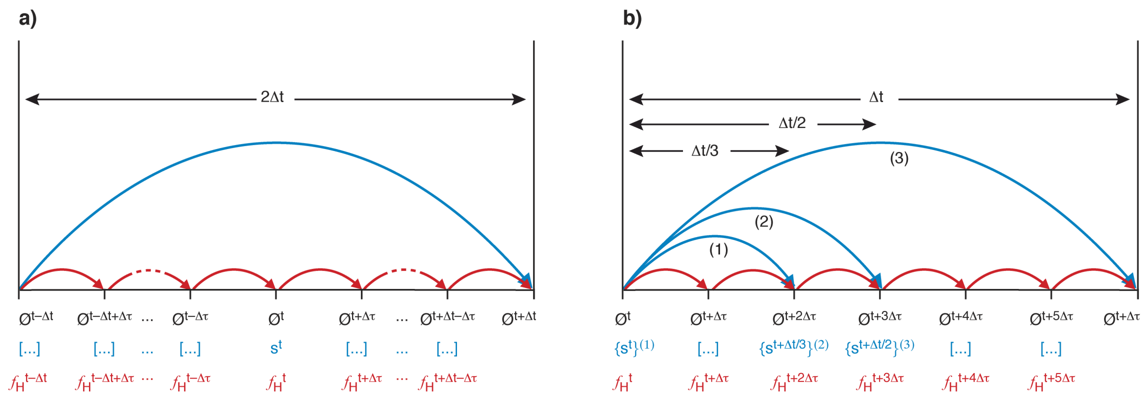

Figure 2: Following Figure 1, illustrations of split-explicit methods for solving the horizontal terms in (5): using (a) leapfrog to solve terms in s between $t-\Delta t$ and $t+\Delta t$; and (b) a 3rd-order Runge-Kutta scheme that evaluates contributions from s between t and $t+\Delta t$ using 2 predictor stages at $t+\Delta t/3$ and $t+\Delta t/2$. Contributions from f_H are updated on sub-steps $\Delta\tau$.

2.2 Recent HEVI proposals

Recent work has proposed a simplification from the split-explicit method: to remove the $\Delta\tau$ sub-stepping, by choosing Δt appropriate for stably resolving contributions from f_H. The result is a HEVI scheme for solving the system

$$\frac{\partial F}{\partial t} = f(F) + s(F), \tag{6}$$

where, now, $f \equiv f_V$ contains all the fast contributions acting in the vertical, and s contains all other terms (including f_H from Eq. (5)). An implicit scheme is used to solve terms in f, which places no stability limit on the length of time-step. But by comparison with the split-explicit approach, it is clear that the time-step required to stably resolve terms in s with an explicit scheme will become $\Delta t \approx \Delta\tau$.

What can be the motivation for moving to a method which *reduces* the length of time-step, thereby increasing the computational cost?

From recent work to develop high-resolution global models, there is a concern that the split-explicit method no longer offers a sufficiently large efficiency gain over a simple HEVI approach (Gassmann, 2013), particularly for models that extend above the stratosphere (up to $O(100\,\text{km})$ altitude) and which, therefore, include the stratospheric polar jet at height $\sim 35\,\text{km}$, which reaches speeds of $O\left(10^2\,\text{m}\,\text{s}^{-1}\right)$ (Kallberg *et al.*, 2005, Fig. D29), i.e. similar to c_s. For this reason, despite their experience with the split-explicit approach in the COSMO model, the Deutscher Wetterdienst (DWD) will move to a non-split HEVI scheme in their next-generation ICON model (see Zängl, 2014, in these proceedings).

In addition, without a gain in efficiency to motivate use of a split-explicit approach, the need for multiple damping operators to stabilise such a scheme (e.g. Baldauf, 2010) is arguably further incentive for finding an alternative.

3 Runge-Kutta schemes for HEVI solutions of the atmosphere

To construct a HEVI scheme for solving (6), we require an implicit scheme for terms in f and an explicit scheme for terms in s *and* a method for combining the schemes. There is a wealth of research in the wider literature into so-called "IMEX" (implicit-explicit) schemes, mostly for use in very stiff diffusion-dominated (parabolic) systems. Such schemes have been less widely analysed in the context of the atmospheric (a hyperbolic) system. However, very recently, some relevant analyses have appeared in the literature: Durran and Blossey (2012); Ullrich and Jablonowski (2012); Giraldo *et al.* (2013); Lock *et al.* (2014); Weller *et al.* (2013). The remainder of this discussion considers the analyses described in Lock *et al.* (2014) and Weller *et al.* (2013).

The analyses focus on Runge-Kutta (RK) methods which use *multiple stages* to integrate over a *single step* (so-called, "single-step, multi-stage" methods): compare the 3-stage RK method used for solving s depicted in Figure 2(b), which integrates over interval Δt, with the leapfrog scheme in Figure 2(a) which operates (in a single stage) over interval $2\Delta t$.

A potential dis-advantage of multi-step methods (such as leapfrog) is that the additional degrees of freedom introduced by using model solutions from multiple time-levels generates computational modes. For some schemes, these computational modes can be inherently well-behaved causing little damage to the physical solution (see e.g. Durran and Blossey, 2012); but some, such as leapfrog, require the addition of a damping operator, which reduces the accuracy of the scheme. By contrast, single-step methods do not support inherent computational modes.

3.1 Defining a Runge-Kutta IMEX scheme

A ν-stage RK IMEX scheme that steps system

$$\mathbf{y}_t = \mathbf{s}(\mathbf{y},t) + \mathbf{f}(\mathbf{y},t),$$

from time $t = n\Delta t$ to $t = (n+1)\Delta t$, by integrating terms in $\mathbf{s}$ with the explicit component of the scheme and $\mathbf{f}$ with the implicit, is described in general by:

$$\begin{aligned}
\mathbf{y}^{(j)} &= \mathbf{y}^n + \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk}\mathbf{s}\left(\mathbf{y}^{(k)}, t+\tilde{c}_k\Delta t\right) + \Delta t \sum_{l=1}^{j} a_{jl}\mathbf{f}\left(\mathbf{y}^{(l)}, t+c_l\Delta t\right), \; j = 1,\nu, \\
\mathbf{y}^{n+1} &= \mathbf{y}^n + \Delta t \sum_{j=1}^{\nu} \tilde{w}_j\mathbf{s}\left(\mathbf{y}^{(j)}, t+\tilde{c}_j\Delta t\right) + \Delta t \sum_{j=1}^{\nu} w_j\mathbf{f}\left(\mathbf{y}^{(j)}, t+c_j\Delta t\right),
\end{aligned} \tag{7}$$

where contributions to the j-th predictor stage of the scheme are described by the elements of matrix $\tilde{\mathscr{A}} = \{\tilde{a}_{jk}\}$ with $\tilde{a}_{jk} = 0$ for $k \geq j$ defining the explicit scheme, and $\mathscr{A} = \{a_{jk}\}$ with $a_{jk} = 0$ for $k > j$ defining the implicit scheme. The new solution $\mathbf{y}^{n+1}$ is a linear combination of the ν predictor stages with weights $\tilde{w}_j$ and w_j (obeying $\sum \tilde{w}_j = 1$ and $\sum w_j = 1$, for $j = 1,\nu$) for the explicit and implicit schemes respectively. (See, e.g., Ascher *et al.*, 1997, for more details.)

Schemes described by (7) can be compactly represented by a "double Butcher tableau",

$$\begin{array}{c|c} \tilde{c} & \tilde{\mathscr{A}} \\ \hline & \tilde{w}^T \end{array} \qquad \begin{array}{c|c} c & \mathscr{A} \\ \hline & w^T \end{array}, \tag{8}$$

where the left-hand tableau describes the explicit component of the RK IMEX scheme, and the right-hand tableau the implicit component; and $\tilde{c}_j = \sum_{k=1}^{j-1} \tilde{a}_{jk}$ and $c_j = \sum_{k=1}^{j} a_{jk}$.

A number of key features of a scheme can be easily identified from the double Butcher tableau description:

- the order of accuracy of the explicit, implicit and combined schemes (Pareschi and Russo, 2005);
- the extent of time-level "splitting": whether the predictor stages keep the implicit and explicit contributions synchronised in time, which can be identified by comparing the $\tilde{c}_j$ and c_j entries (Lock *et al.*, 2014);

- whether the final stage is an implicit operation, ensuring a balanced final solution (Ascher *et al.*, 1997); and
- whether the explicit scheme is "strong-stability preserving" (SSP), which guarantees stability (within some restricted time-step) for non-smooth solutions (Gottlieb *et al.*, 2001).

3.2 Analyses of RK IMEX schemes for HEVI solutions

In Lock *et al.* (2014) and Weller *et al.* (2013), RK IMEX schemes are identified from the literature for analysis in the context of HEVI solutions of atmospheric equations. The studies include a linear analysis of the schemes applied to a system supporting acoustic waves to consider errors in amplitude and phase representation, by a von Neumann stability analysis; and consideration of the errors and convergence rates from the different schemes when applied to an idealised numerical experiment that generates acoustic and gravity waves.

Weller *et al.* (2013) identify two alternative HEVI solutions of the acoustic waves:

- the "UFPreF" formulation, whereby all terms acting in the horizontal are solved explicitly and all terms acting in the vertical are solved implicitly; and
- the "UFPreB" formulation, which recognises that during a sequential numerical computation of model variables, the updated horizontal wind-components (u) will be available for use in computing the new pressure p, and could therefore contribute according to the implicit tableau but without requiring any new computations.

The names "UFPreF" ("U-Forward/P-Forward") and "UFPreB" ("U-Forward/P-Backward") make allusion to the "forward-backward" method of Mesinger (1977), used to solve the horizontally propagating fast waves on the sub-steps in split-explicit models (see Sect. 2.1), where the fast terms contributing to the new u solution are solved by a forward Euler step, which then contribute to the new p solution according to a backward Euler step, enhancing stability.

The linear analyses in Lock *et al.* (2014) consider amplitude and phase errors from the schemes for a range of Courant numbers that describe scales associated with next-generation high-resolution NWP forecasts ($\Delta x = \mathrm{O}\,(1\,\mathrm{km})$) *and* coarse-resolution climate simulations ($\Delta x = \mathrm{O}\,(100\,\mathrm{km})$), with the aim of identifying schemes for use in a "seamless" modelling approach.

Of the schemes analysed, two perform consistently well across all analyses: "Trap2(2,3,2)" (from Wood *et al.*, 2013) and "UJ3(1,3,2)" (described as "Strang carryover" in Ullrich and Jablonowski, 2012). Both schemes are 2nd-order accurate; yield a relatively wide stability limit that is determined by the explicit scheme and not reduced by combination with the implicit scheme acting on large vertical Courant numbers; finish with an implicit stage ensuring balance in the new model solution; and include an SSP explicit RK scheme. Both schemes offer good accuracy in terms of amplitude and phase errors in both UFPreF and UFPreB formulation, which offers flexibility in their application. In addition, Trap2(2,3,2) avoids time-level splitting of the explicit and implicit contributions at each integration stage; and is estimated to be the most cost-effective in terms of computation time (due to its relatively wide stability window and few stages). A brief summary of the stability and cost of the schemes considered in Lock *et al.* (2014) is presented in Table 1.

In Weller *et al.* (2013), the schemes are applied to a non-linear, near-hydrostatic Boussinesq test from Durran and Blossey (2012), which uses a localised streamfunction forcing to generate gravity waves in a stratified shear flow. Following Durran and Blossey (2012), results from the experiments are presented in terms of RMS errors in the buoyancy field for varying Δt with respect to a 4th-order explicit RK scheme run with a very small time-step ($\Delta t = 0.5\,\mathrm{s}$). For very large Δt, the schemes are applied in

Table 1: Indication of schemes' computational costs (from Lock et al., 2014): v and s indicate the number of RHS computations (total and implicit respectively) required for a single timestep; $\Delta t^*_{\max}$ is the largest horizontal Courant number which ensures stability for all vertical Courant numbers; and $v/\Delta t^*_{\max}$ indicates relative cost. Numbers are included for UFPreF and UFPreB solutions of the acoustic system. For details of the schemes, see Lock et al. (2014).

	Number of RHS evaluations		$\Delta t^*_{\max}$		$v/\Delta t^*_{\max}$	
Scheme	Total (v)	Implicit (s)	UFPreF	UFPreB	UFPreF	UFPreB
Trap2(2,3,2)	3	2	2.00	2.00	1.5	1.5
UJ3(1,3,2)	4	1	1.73	2.00	2.31	2.00
SSP3(3,3,2)	4	3	1.70	2.48	2.35	1.61
SSP3(4,3,3)	5	4	0.89	1.39	5.62	3.60
ARK2(2,3,2)	3	2	1.25	>2.5	2.40	<1.20
ARS3(2,3,3)	3	2	1.19	0.00	2.52	∞
ARS3(4,4,3)	4	4	1.54	0.00	2.60	∞

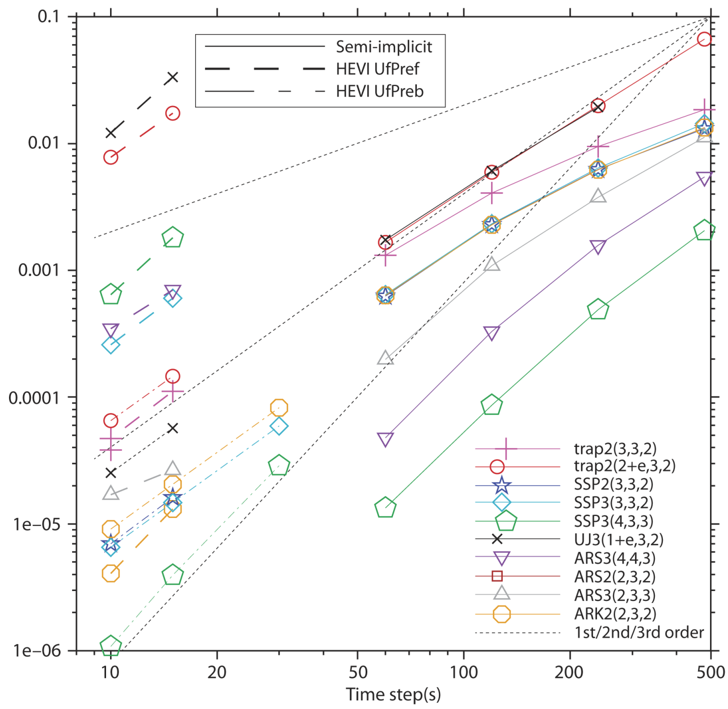

Figure 3: RMS buoyancy errors for the test case described in Durran and Blossey (2012) for semi-implicit (solid), HEVI UFPreF (dash) and HEVI UFPreB (dot-dash) solutions. Reproduced from Weller et al. (2013, Fig. 4) — for full details of the experiment and schemes, refer to paper.

"semi-implicit" form, i.e. both vertical *and* horizontal acoustic and gravity modes are solved with the implicit component of the IMEX scheme; while for Δt small enough, the schemes are applied in HEVI form. The study considers overall convergence rates; schemes whose convergence rate is not reduced in HEVI form relative to semi-implicit; and absolute errors. Results are reproduced in Figure 3.

The two schemes identified as performing best in Lock *et al.* (2014) — Trap2(2,3,2) (indicated in Figure 3 by red markers) and UJ3(1,3,2) (black markers) — are shown to perform well for UFPreB solutions: showing continued 2nd-order convergence rates from the semi-implicit implementation; but they lose accuracy for UFPreF. In addition, the analysis identifies the scheme "ARK2(2,3,2)" (Giraldo *et al.*, 2013) (indicated by yellow markers), which also exhibits 2nd-order convergence, but with lower absolute errors and is the only scheme showing good performance for both UFPreF and UFPreB.

4 Summary

Based on the assumption that numerical schemes that minimise the use of non-local data are most desirable for future massively parallel computer architectures, this paper considers the use of time-stepping schemes that solve all horizontally propagating modes with an explicit scheme, and only use an implicit scheme for fast-moving vertical modes — so-called "HEVI" schemes. The most widespread HEVI schemes in current models are "split-explicit" schemes, which integrate the fastest horizontal contributions on a short sub-step — a short review of such schemes is presented. More recently, some modelling groups have moved to a simpler HEVI approach which avoids sub-stepping, arguing that there is little efficiency gain from the split-explicit approach for models that include representation of the high-speed stratospheric polar-night jet. Consequently, there have been several recent papers proposing and analysing new HEVI schemes for use in atmospheric models. The results of two such papers (Lock *et al.*, 2014; Weller *et al.*, 2013) are summarised here.

From linear analyses of a simple atmospheric system comprising the terms that pose the greatest stability constraint and combined with numerical experimentation, the two studies consider stability, accuracy and cost of several Runge-Kutta (RK) implicit-explicit (IMEX) schemes implemented in two HEVI forms: "UFPreF" (the most intuitive form on paper) and "UFPreB" (a variation that exploits the most up-to-date solutions at each computation stage). The results demonstrate that, in general, RK HEVI solutions can offer suitable stability and accuracy; and that UFPreB formulations generally enhance stability. The linear analyses identify "Trap2(2,3,2)" and "UJ3(1,3,2)" as performing consistently well by several measures: they yield good stability in the explicit limit, which is not reduced by the use of an implicit solver for the vertical fast waves; amplitudes and phase are well-represented across the phase-space for both UFPreF and UFPreB solutions; both offer relatively good efficiency, defined as accuracy for a given cost. In the numerical experiments, both schemes perform well in UFPreB, though less well in UFPreF; and a third scheme "ARK(2,3,2)" is shown to offer similarly small errors, *and* is the only scheme tested, which keeps small errors for *both* UFPreF and UFPreB solutions.

It is interesting to note that of the schemes analysed in Lock *et al.* (2014) and Weller *et al.* (2013), the three identified as performing best — "Trap2(2,3,2)", "UJ3(1,3,2)" and "ARK2(2,3,2)" — have all been developed for atmospheric models. Meanwhile, RK IMEX schemes identified from the wider literature, mostly designed for advection-diffusion problems, perform less well: the high-order SSP schemes are shown to offer good accuracy, but with relatively poor stability and high cost; and the non-SSP schemes, albeit designed for systems with non-smooth solutions, are inherently unstable for UFPreB solutions of the atmospheric system.

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