Some issues in numerical stochastic weather/climate modeling

or

How do I use Stochastic Differential Equations to model something real?

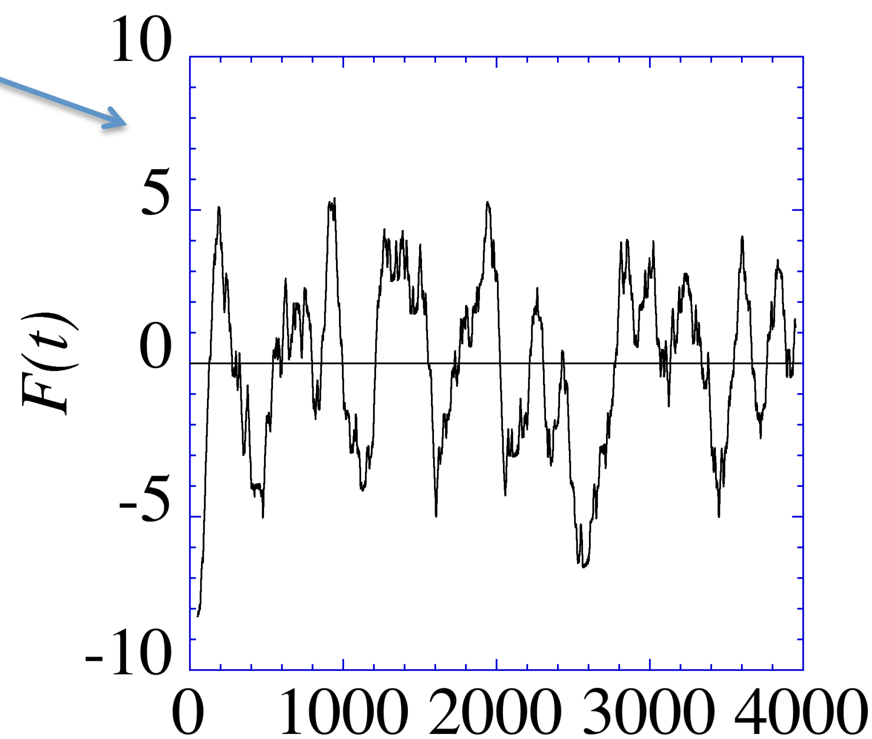
Cécile Penland, with thanks to Prashant Sardeshmukh, Roger Témam, Brian Ewald, James A. Hansen, and many others.

Outline

- Schematic
- A Theorem
- Stochastic Taylor expansion (leading to)
- Some integration schemes
- **Examples, including cautionary tales**
- Some ways to beat the system
- List of useful references

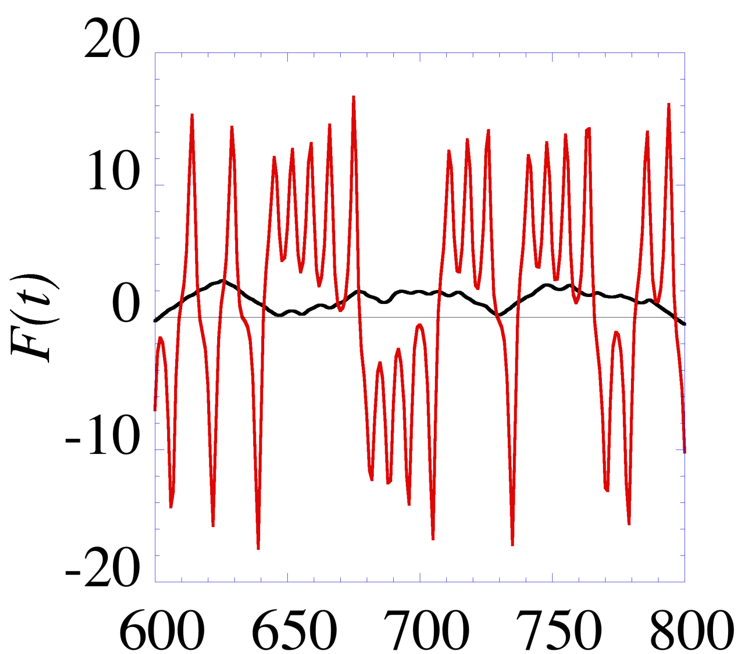
Let’s say we have a simple system:

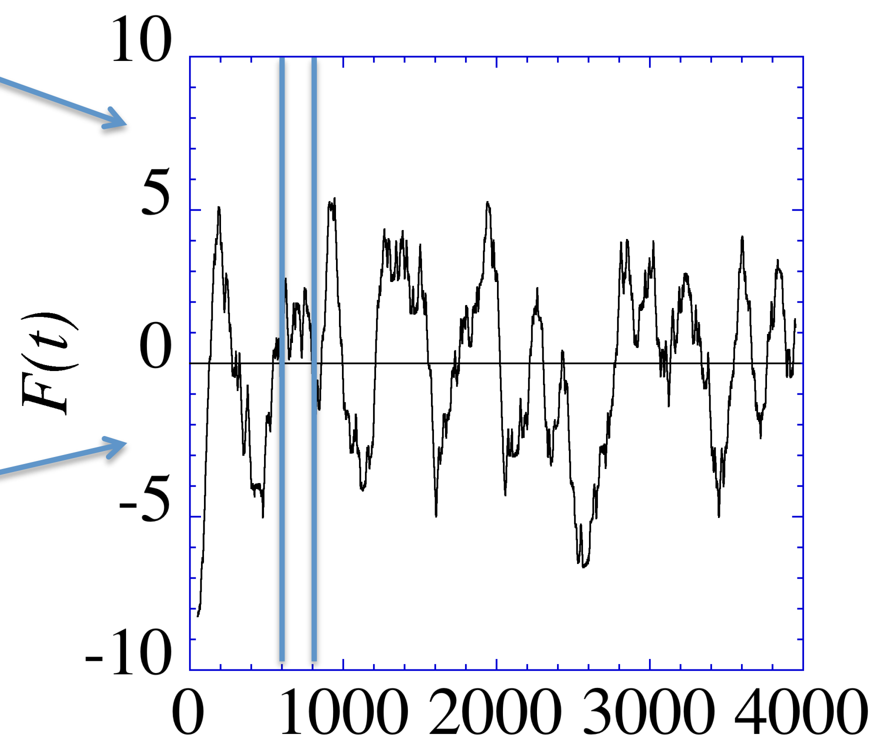
$$\frac{dx}{dt} = G(x) + F(t)$$



But our simple system may not be so simple at a finer timescale:

$$\frac{dx}{dt} = G(x) + F(t)$$





Papanicolaou and Kohler (1974)

Choose a scaling $s = \varepsilon^2 t$:

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{G}(\boldsymbol{x}, s/\varepsilon^2) + \frac{1}{\varepsilon}\boldsymbol{F}(\boldsymbol{x}, s/\varepsilon^2) \quad (*)$$

For simplicity, say

$$F_i(\boldsymbol{x}, s/\varepsilon^2) = \sum_k F_i^k(\boldsymbol{x}, s)\eta_k(s/\varepsilon^2)$$

and

$$C_{km} = \int_{-\infty}^{\infty} < \eta_k(t)\eta_m(t'+t) > dt' \equiv (\phi\phi^T)_{km}$$

$$\lim_{\substack{t \to \infty \\ \varepsilon \to 0}} (*) \to \quad d\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{x}, s)\, ds + \sum_{k,\alpha} \boldsymbol{F}^k(\boldsymbol{x}, s)\phi_{k\alpha} \bullet dW_\alpha$$

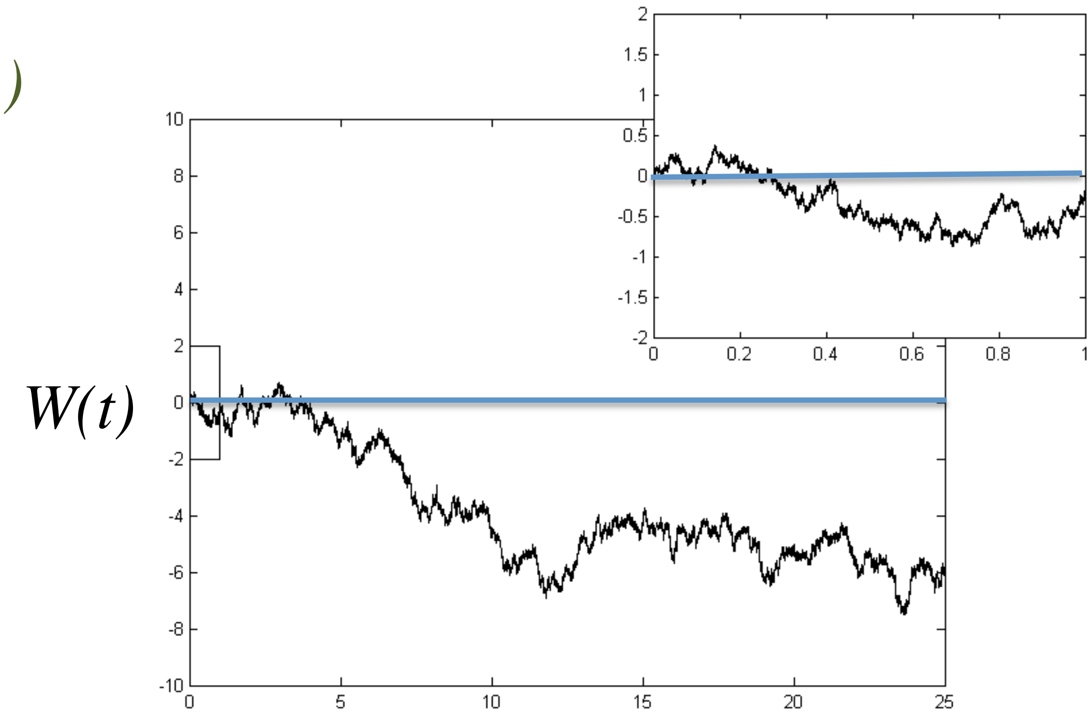
(W is a Brownian motion; $dW \in \boldsymbol{N}(0, dt)$).

Wiener process, or Brownian motion

- $<W(t)>=0$
- $<W(t)W(t')>=\min(t,t')$
- $<dW(t)dW(t')>=dt\,\delta(t-t')$
- Two sets of integration rules found in nature.

$$\int_0^t W\,dW = W^2(t) - \frac{t}{2}$$

$$\int_0^t W \bullet dW = W^2(t)$$



And now: Back to Calculus 101!

Deterministic system:

$$\frac{dX(t)}{dt} = f(X)$$

$$\frac{dF(X)}{dt} = \frac{dF}{dX}\frac{dX(t)}{dt} = f(X)\frac{dF}{dX}$$

Just rewriting:

$$dF(X) = LF(X)dt, \qquad L = f(X)\frac{d}{dX}$$

Through an iterative procedure we get

$$F(X(T)) = F(X(0)) + \boxed{LF(X(0))T} + \boxed{\frac{1}{2}L^2F(X(0))T^2} + \ldots$$

The stochastic version:

$$dX = f(X,t)dt + g(X,t)(\bullet)dW.$$

Define:

$$L^o = \frac{\partial}{\partial t} + f(X,t)\frac{\partial}{\partial X} \qquad \text{(Stratonovich)}$$

$$L^o = \frac{\partial}{\partial t} + f(X,t)\frac{\partial}{\partial X} + g^2(X,t)\frac{1}{2}\frac{\partial^2}{\partial X^2} \qquad \text{(Ito)}$$

And for either case:

$$L^1 = g(X,t)\frac{\partial}{\partial X}$$

Get the stochastic Taylor expansion by iterating

$$F(X(T),T) = F(X(0),0) + \boxed{\int_0^T L^1 F(X(t),t)(\bullet)\,dW} + \boxed{\int_0^T L^o F(X(t),t)\,dt}$$

We develop integration schemes by identifying $F(X) = X(t)$.

Rules of order: Depends on whether of system is to be integrated in sense of Ito or Stratonovich.

Ito: Count 1 for every integral over time and 1/2 for every integral over dW, unless term has only time (i.e., no dW) in it. In that case, subtract 1/2 from what it would otherwise be.

Stratonovich: only schemes of total integer orders are valid.

So now we want to integrate

$$dX = f(X,t)dt + g(X,t)(\bullet)dW.$$

Lowest order (0.5): Stochastic Euler scheme: $(\Re \in \mathbf{N}(0,1))$.

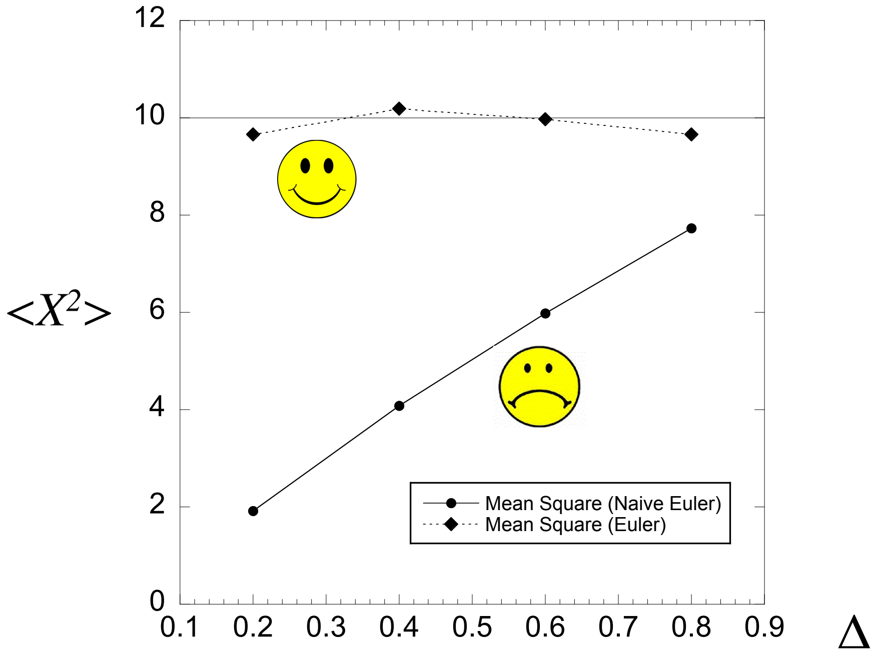
$$X(t_{n+1}) = X(t_n) + f(X_n,t)\,\Delta + g(X_n,t)(\Delta W),$$

where $\Delta W = \Re\sqrt{\Delta}$.

Time for a cautionary tale:

$$\frac{dX(t)}{dt} = -\gamma X(t) + \sigma\xi$$

Let's compare:



$$X(t_{n+1}) = X(t_n) - \gamma X(t_n)\Delta + \sigma\, \mathfrak{R}\sqrt{\Delta}\ . \quad \text{(Euler)}$$

vs

$$X(t_{n+1}) = X(t_n) + (-\gamma X(t_n) + \sigma\, \mathfrak{R})\, \Delta \quad \text{(Naïve Euler)}$$

Again we want to integrate

$$dX = f(X,t)dt + g(X,t)(\bullet)dW.$$

This time we'll use an explicit O(1) Mil'steyn scheme: $(\Re \in \mathbf{N}(0,1))$.

$$X(t_{n+1}) = X(t_n) + f(X,t_n)\Delta + g(X,t_n)\Delta W$$

$$+g(X,t_n)\frac{\partial g(X,t_n)}{\partial X} I_{(1,1)} \qquad \Delta W = \Re\sqrt{\Delta}.$$

Stratonovich: $I_{(1,1)} = (\Delta W)^2/2$

Ito: $I_{(1,1)} = [(\Delta W)^2 - \Delta]/2$

Comments: Multivariate system more complicated; use same variate; etc.

Time for another cautionary tale:

$$dX(t) = \mathbf{L}X(t)dt + \mathbf{S}dW$$

$$< XX^T >= \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

```
 do i = 1,n
  r(i)=gasdev(idm)*sqrt(dt)
 enddo
c
 do i = 1,n
   x(i)=x0(i)
   do j = 1,n
      x(i)=x(i)+L(i,j)*dt
.         +S(i,j)*r(j)
   enddo
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$$< XX^T >= \begin{pmatrix} 1.98 & -.01 & .99 & .00 \\ -.01 & 1.99 & .02 & 1.00 \\ .99 & .02 & 2.02 & -.01 \\ .00 & 1.00 & -.01 & 2.01 \end{pmatrix}$$

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What about implicit or semi-implicit schemes?

- Implicit Euler
- Implicit Mil’steyn
- Implicit Ewald-Témam

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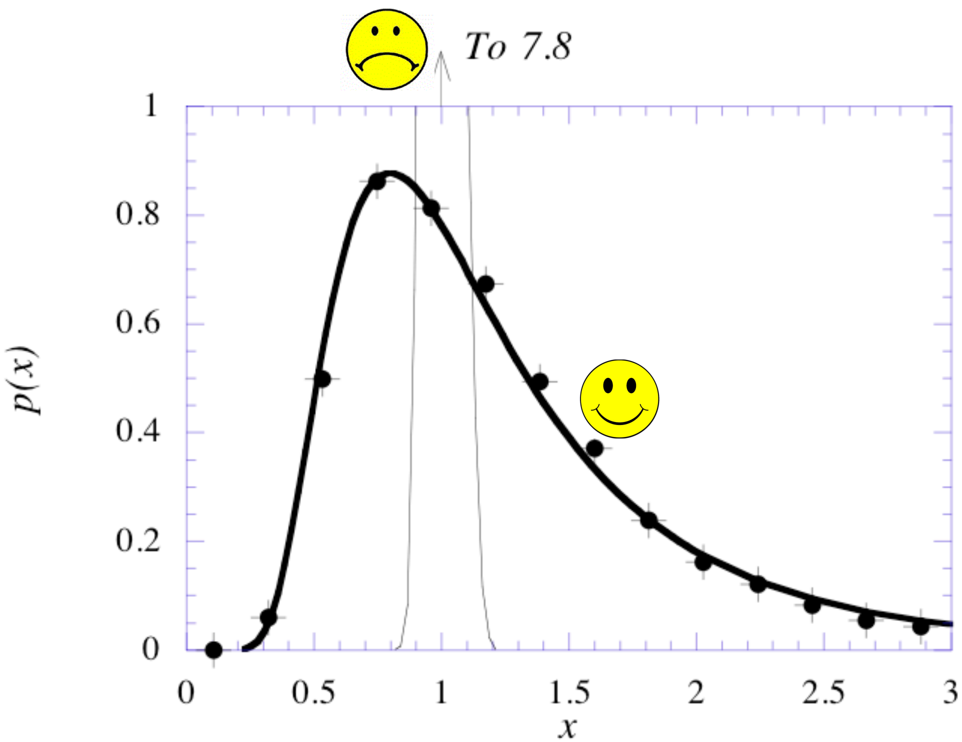
Never, ever, put a random number into the denominator!

Yet another cautionary tale:

$$\frac{dx}{dt} = (k^2 - r)x + F \qquad r = r_o + \eta\xi, \quad \xi \text{ stochastic}$$



Heavy line: Analytic solution

Dots: Implicit Ewald-Témam

Light line: Naïve implicit scheme:

$x'(t+2\Delta) = x(t) + 2F\Delta$

$x(t+2\Delta) = x'(t+2\Delta) + 2\Delta\,[k^2-(r_o+\eta\xi)]\,x(t+2\Delta)$

$$x(t+2\Delta) = [x(t) + 2F\Delta] \;/\; \{1-2\Delta\,[k^2-(r_o+\eta\xi)]\}$$

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Sometimes.

Can't I just stochastify a few parameters without rewriting the whole model?

In explicit schemes such as Runge-Kutta, you can often get away with using deterministic code by replacing the amplitude σ of the noise with $\sigma/\sqrt{\Delta}$.

If your timestep Δ is so small that it's smaller than *any* dynamical timescale in your system, you may not have to inject the randomness every timestep. If I inject the noise every N timesteps, I *might* get away with deterministic code by replacing the amplitude σ of the noise with $\sigma\sqrt{N\Delta}$. Useful if σ is diagnosed through data assimilation with assimilation window = $N\Delta$.

Caveat Emptor!

Some References
(Also see references therein.)

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