Overview of the Numerics of the ECMWF Atmospheric Forecast Model

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Characteristics of the ECMWF model

- Hydrostatic shallow-atmosphere approximation
- Pressure-based hybrid vertical coordinate
- Two-time-level semi-Lagrangian semi-implicit time integration scheme
- Spectral horizontal representation (spherical harmonics)
- Pseudo-spectral (finite-element) vertical representation
- Transform method for computing non-linear terms using non-staggered grid
- Fourth order horizontal diffusion

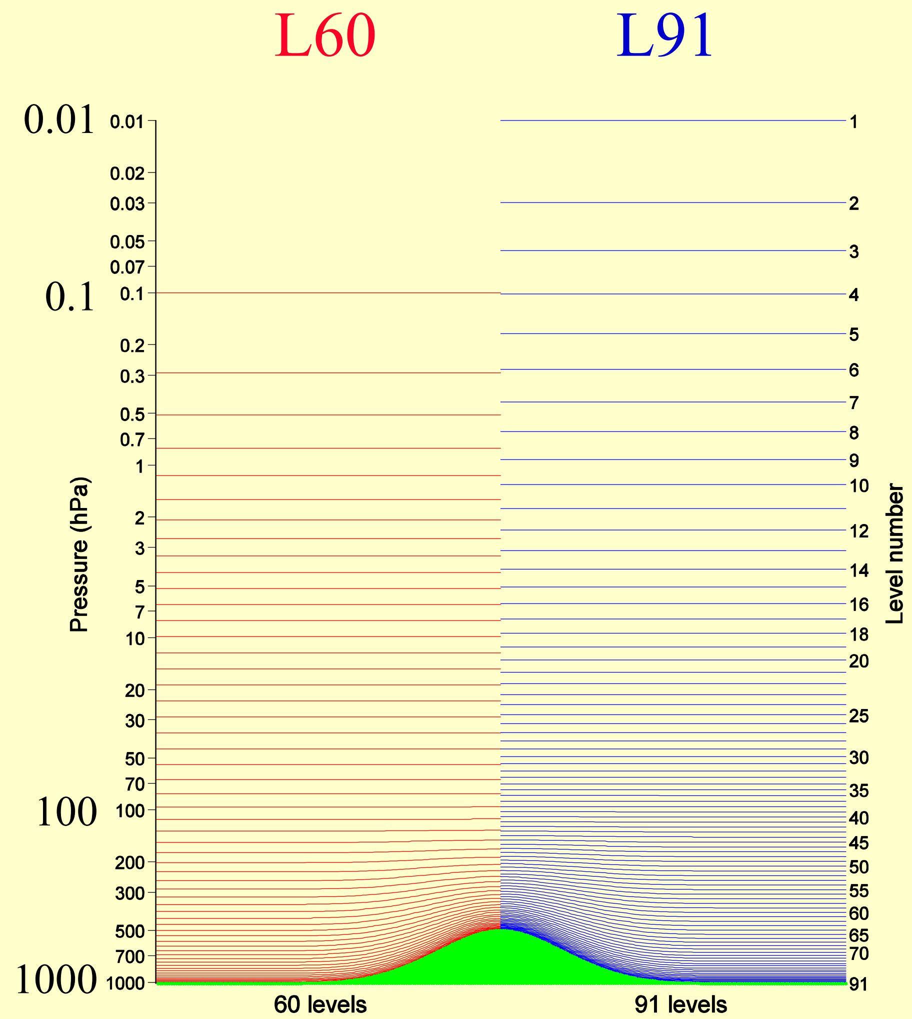
Vertical coordinate

Pressure-based hybrid coordinate 0

$$
\begin{cases} p(\eta = 0) = 0 \\ p(\eta = 1) = p_s \end{cases}
$$

$$
\frac{\partial p}{\partial \eta} = \frac{dA(\eta)}{d\eta} + \frac{dB(\eta)}{d\eta} p_s
$$

$$
\int_0^1 \frac{dA}{d\eta} d\eta = 0 \; ; \quad \int_0^1 \frac{dB}{d\eta} d\eta = 1
$$



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Model equations

- **Momentum**

pressure-gradient

$$\frac{d\vec{V}_h}{dt} = -f\vec{k}\times\vec{V}_h \underbrace{-\nabla_h\phi - R_d T_v \nabla_h \ln p}_{\text{pressure-gradient}} + P_V + K_V$$

Discretized in vector form to avoid pole problems

- **Thermodynamics**

$$\frac{dT}{dt} = \frac{\kappa T_v \omega}{(1+(\delta-1)q)p} + P_T + K_T$$

- **Hydrostatic**

$$\phi = \phi_s - \int_1^{\eta} R_d T_v \frac{\partial}{\partial\eta}(\ln p)\, d\eta$$

$\vec{V}_h$: horizontal wind vector

T_v : virtual temperature

∇_h : "horizontal" gradient

ω : p − coordinate vertical velocity

$\kappa \equiv R_d / c_{pd}$, $\delta \equiv c_{pv} / c_{pd}$

P_V, P_T : physical parameterization

K_V, K_T : horizontal diffusion

Model equations (cont)

- **Continuity equation**

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right)+\nabla_h \cdot\left(\vec{V}_h \frac{\partial p}{\partial \eta}\right)+\frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)=0$$

- **Humidity equation**

$$\frac{dq}{dt}=P_q$$

- **Ozone equation**

$$\frac{dr_{o_3}}{dt}=P_{o_3}$$

Vertical integration of the continuity equation

$$\frac{d}{dt}(\ln p_s) = \int_0^1 \left(\frac{dB}{d\eta}\frac{\partial}{\partial t}(\ln p_s) + \frac{dB}{d\eta}\vec{V}_h \cdot \nabla \ln p_s\right) d\eta$$

where

$$\frac{\partial}{\partial t}(\ln p_s) = -\frac{1}{p_s}\int_0^1 \nabla \cdot \left(\vec{V}_h \frac{\partial p}{\partial \eta}\right) d\eta$$

$$\omega = -\int_0^\eta \nabla \cdot \left(\vec{V}_h \frac{\partial p}{\partial \eta}\right) d\eta + \vec{V}_h \cdot \nabla p$$

Needed for the energy-conversion term in the thermodynamic eq.

$$\dot{\eta}\frac{\partial p}{\partial \eta} = -\frac{\partial p}{\partial t} - \int_0^\eta \nabla \cdot \left(\vec{V}_h \frac{\partial p}{\partial \eta}\right) d\eta$$

Needed for the semi-Lagrangian trajectory computation

Vertical integration (finite elements)

$$F(\eta) = \int_0^\eta f(x)dx$$

can be approximated as $\sum_{i=K_1}^{K_2} C_i d_i(\eta) \approx \sum_{i=M_1}^{M_2} c_i \int_0^\eta e_i(x)dx$

Basis sets

and, applying the Galerkin method:

$$\sum_{i=K_1}^{K_2} C_i \int_0^1 t_j(x) d_i(x) dx = \sum_{i=M_1}^{M_2} c_i \int_0^1 \left(t_j(x) \int_0^x e_i(y)\, dy \right) dx \qquad \text{for } N_1 \le j \le N_2$$

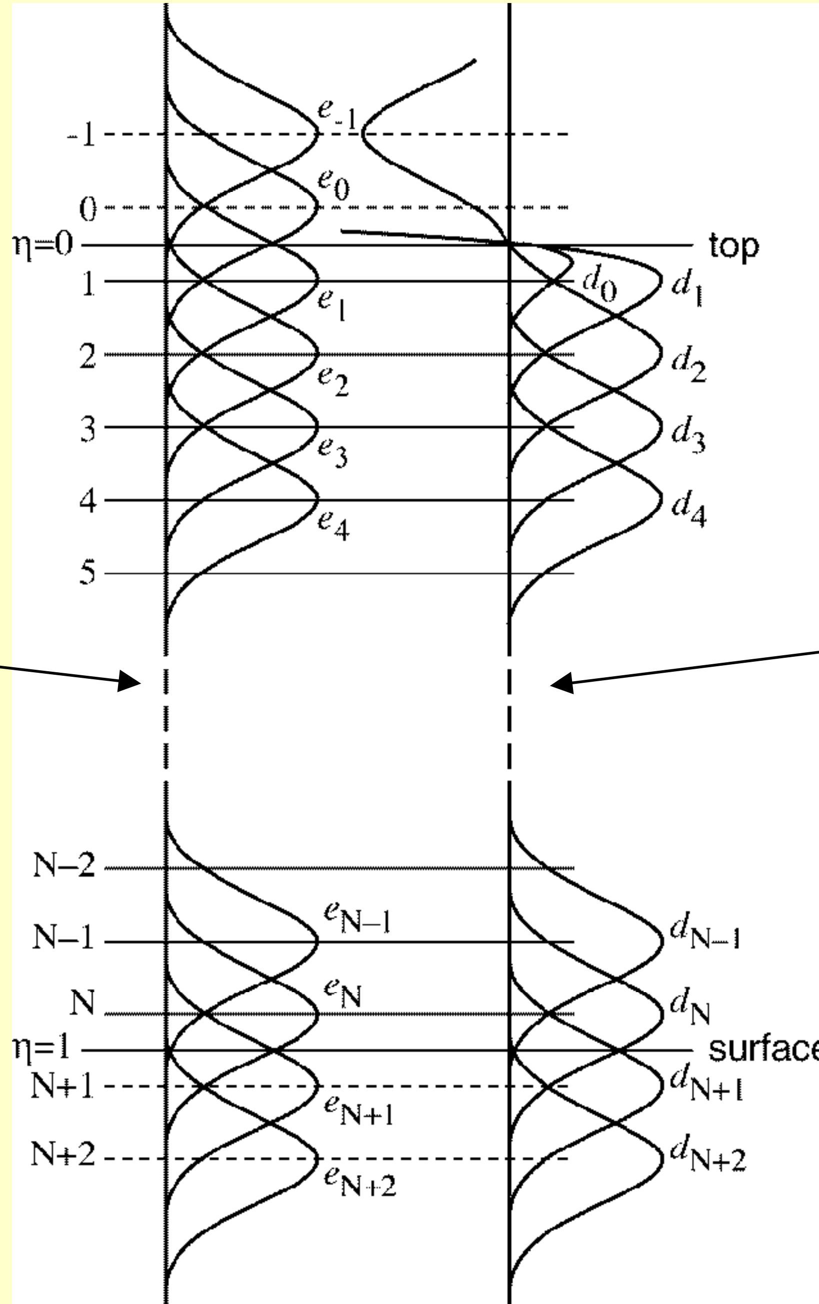
$$\underline{\underline{\mathbf{A}}}\vec{\mathbf{C}} = \underline{\underline{\mathbf{B}}}\vec{\mathbf{c}} \Rightarrow \vec{\mathbf{C}} = \underline{\underline{\mathbf{A}}}^{-1}\underline{\underline{\mathbf{B}}}\vec{\mathbf{c}}$$

$$\vec{\mathbf{c}} = \underline{\underline{\mathbf{S}}}^{-1}\vec{\mathbf{f}} \qquad \vec{\mathbf{F}} = \underline{\underline{\mathbf{P}}}\vec{\mathbf{C}}$$

$$\vec{\mathbf{F}} = \underline{\underline{\mathbf{P}}}\,\underline{\underline{\mathbf{A}}}^{-1}\underline{\underline{\mathbf{B}}}\,\underline{\underline{\mathbf{S}}}^{-1}\vec{\mathbf{f}} \equiv \underline{\underline{\mathbf{J}}}\,\vec{\mathbf{f}}$$

Cubic B-splines for the vertical discretization

Basis elements for the represen-tation of the function to be integrated (integrand) f



Basis elements for the representation of the integral F

Advantages of the finite-element scheme in the vertical

- **8th order accuracy using cubic basis functions**
  - ➔ Very accurate computation of the pressure-gradient term in conjunction with the spectral computation of horizontal derivatives
  - ➔ More accurate vertical velocity for the semi-Lagrangian trajectory computation
    - **Improved ozone conservation**
- **Reduced vertical noise in stratosphere**
- **Smaller error in the computation of the integrals than using the finite-element scheme in differential form**
  (Private communication by Staniforth & Wood)

$$f = \frac{\partial F}{\partial \eta}$$

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The spectral horizontal representation

$$X(\lambda,\mu,\eta,t) = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} X_n^m(\eta,t) Y_n^m(\lambda,\mu)$$

Triangular truncation (isotropic)

Spherical harmonics

$$X_n^m(\eta,t) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} X(\lambda,\mu,\eta,t) \overbrace{P_n^m(\mu) e^{-im\lambda}} d\lambda d\mu$$

Fourier functions

Legendre polynomials

Legendre transform
by Gaussian quadrature
using NL ≥ (2N+1)/2
"Gaussian" latitudes (linear grid)
((3N+1)/2 if quadratic grid)
No "fast" algorithm available

FFT using
NF ≥ 2N+1
points (linear grid)
(3N+1 if quadratic grid)

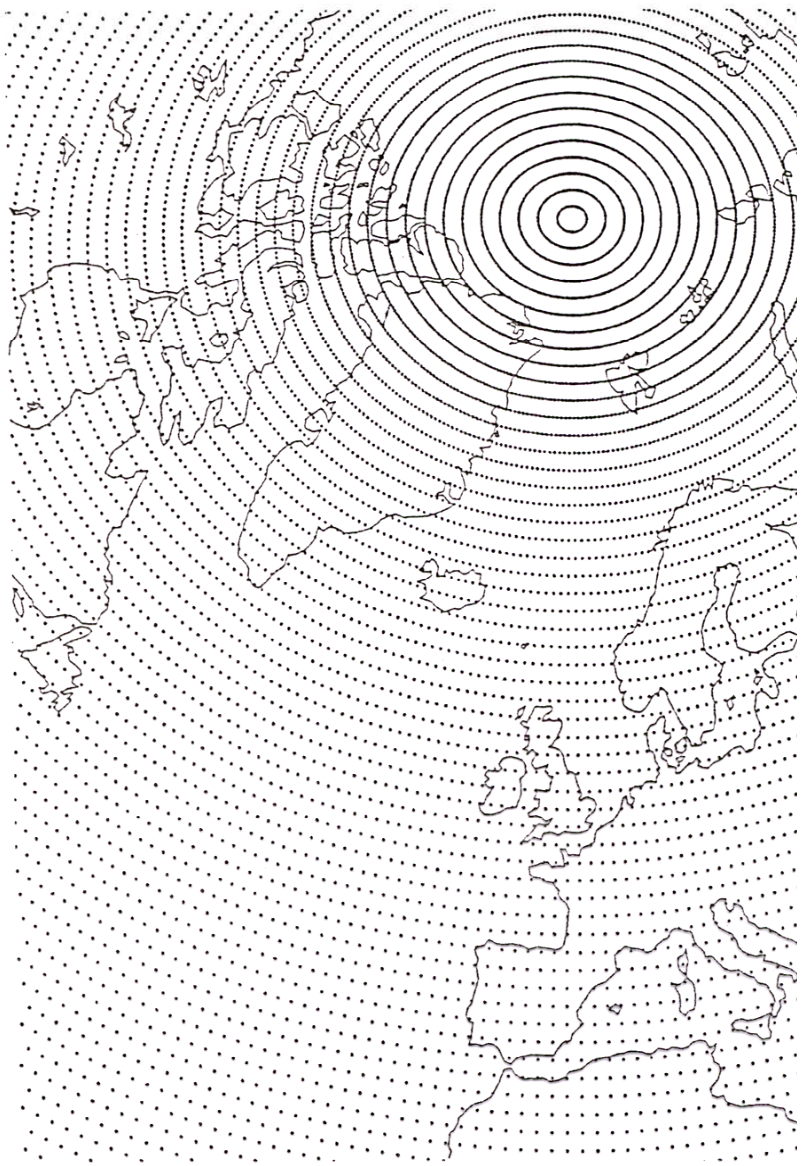
Advantages of the spectral method

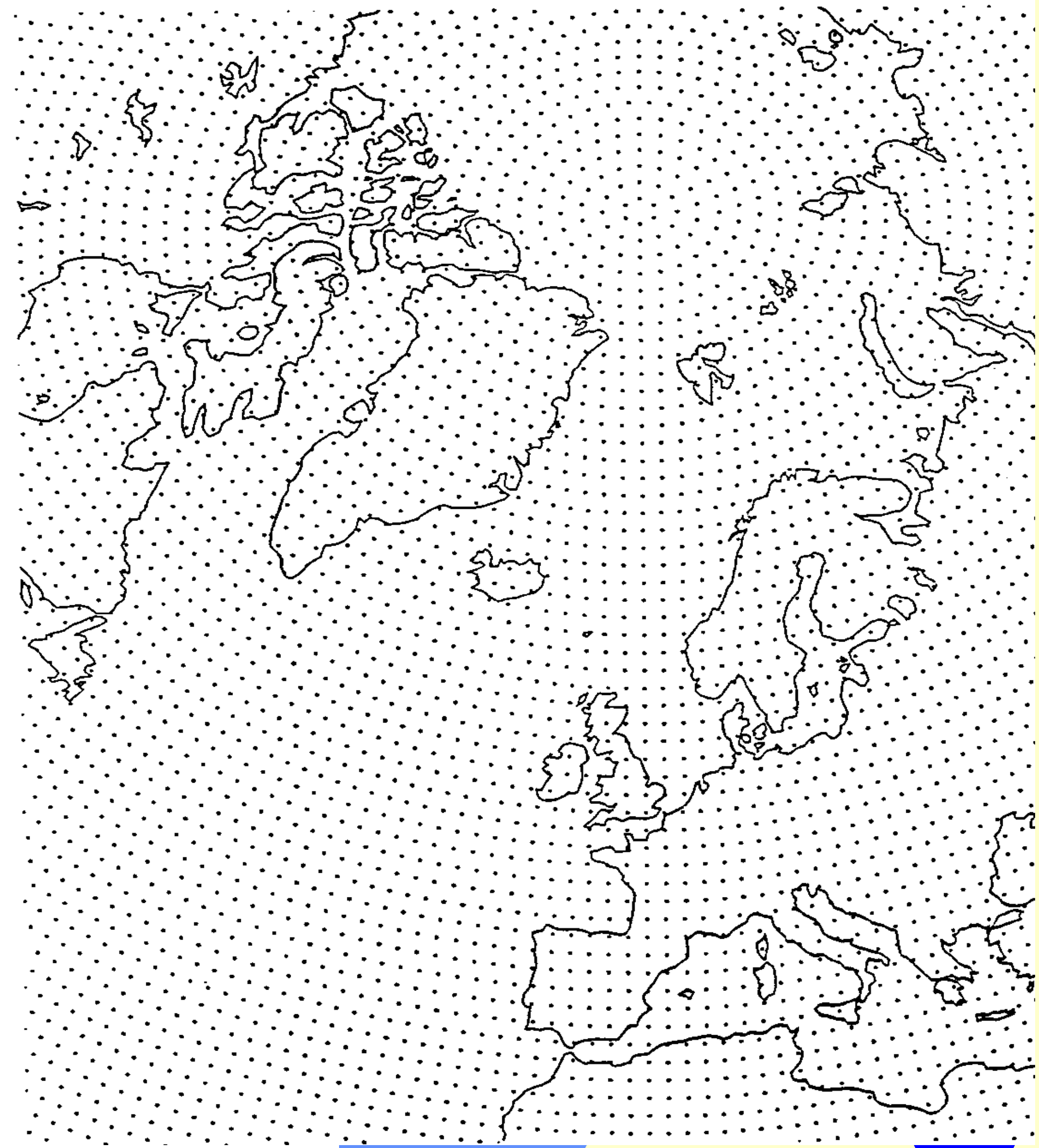
- **Spherical harmonics are eigenfunctions of the Laplace operator:**

$$\nabla^2 Y_n^m = -\frac{n(n+1)}{a^2} Y_n^m$$

  - ➔ The solution of a Helmholtz equation is straightforward
  - ➔ The application of a high-order diffusion is very easy
- **Horizontal derivatives are computed analytically**
  - ➔ The computation of the pressure-gradient term is very accurate
- **But:**
  - ➔ The computational cost of the Legendre transforms increases more rapidly with resolution than the rest of the model

The full and the reduced Gaussian grids





Semi-implicit time integration

Define $\Delta_{tt} X \equiv 0.5\left(X^{+} + X^{-}\right) - X^{0}$

$$\frac{d\vec{V}}{dt} = RHS_V + \Delta_{tt}\left\{\overbrace{\underset{\approx}{\gamma}\nabla_h T + R_d T_r \nabla_h\left(\ln p_s\right)}\right\}$$

Linearized pressure-gradient using a reference temperature T_r and a reference surface pressure $(p_s)_r$

$$\frac{dT}{dt} = RHS_T + \Delta_{tt}\left(\underset{\approx}{\tau} D\right)$$

$$(\underset{\approx}{\gamma} X)_\eta = -\int_1^\eta R_d X \frac{d}{d\eta'}\left(\ln p_r\right) d\eta'$$

$$\frac{d}{dt}(\ln p_s) = RHS_p + \Delta_{tt}\left(\underset{\sim}{\nu} D\right)$$

$$(\underset{\approx}{\tau} X)_\eta = \kappa T_r \left(\frac{1}{p_r}\right)_\eta \int_0^\eta X \frac{dp_r}{d\eta'} d\eta'$$

$$\underset{\sim}{\nu} X = \frac{1}{(p_s)_r}\int_0^1 X \frac{dp_r}{d\eta} d\eta$$

=> $$\left(\underset{\approx}{\mathbf{I}} + \underset{\approx}{\mathbf{\Gamma}}\nabla_h^2\right) D^{+} = \tilde{D}$$

Vertically coupled set of Helmholtz equations

$$\underset{\approx}{\mathbf{\Gamma}} \equiv \underset{\approx}{\gamma}\underset{\approx}{\tau} + R_d T_r \underset{\sim}{\nu}$$

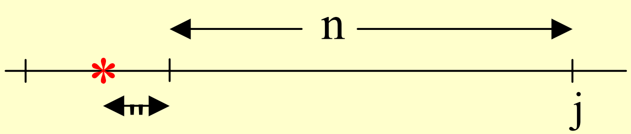
Semi-Lagrangian advection (1)

1D advection equation without RHS:

$$\frac{d\varphi}{dt} = 0 \Rightarrow \frac{\varphi(x_j, t+\Delta t) - \varphi(x_*, t)}{\Delta t} = 0$$

In the Lagrangian point of view, time is the only independent variable (position should be consistent with time)

$$\varphi(x_j, t+\Delta t) = \varphi(x_*, t)$$



Stability analysis:
absolutely stable if the value of $n(x_*,t)$ is computed by interpolation using the surrounding grid points.

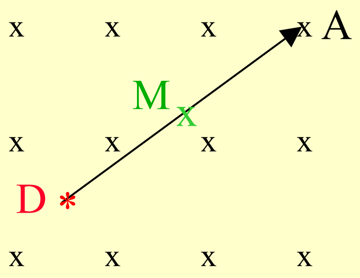
Finding the departure point x_* in the linear case:

$$\frac{dx}{dt} = U_0 \Rightarrow \frac{x_j - x_*}{\Delta t} = U_0$$

$$x_* = x_j - U_0 \Delta t$$

Semi-Lagrangian advection (2)

Three time level scheme with RHS :



$$\frac{\varphi^{A}(t+\Delta t)-\varphi^{D}(t-\Delta t)}{2\Delta t}=R^{M}(t)$$

Centered second order accurate scheme

Disadvantages of three-time-level schemes:

- Less efficient than two-time-level schemes
- Computational mode

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Semi-Lagrangian advection (3)

Two time level second order accurate schemes :

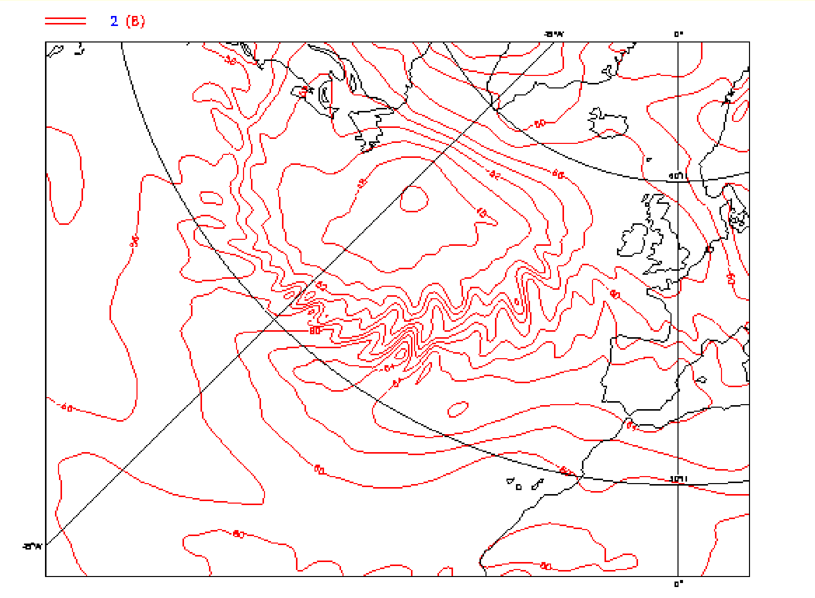
$$\frac{\varphi^A(t+\Delta t)-\varphi^D(t)}{\Delta t}=R^M(t+\frac{\Delta t}{2})$$

with

$$R(t+\frac{\Delta t}{2})\approx\frac{3}{2}R(t)-\frac{1}{2}R(t-\Delta t)$$

Unstable

Forecast 200 hPa T
from 1997/01/04



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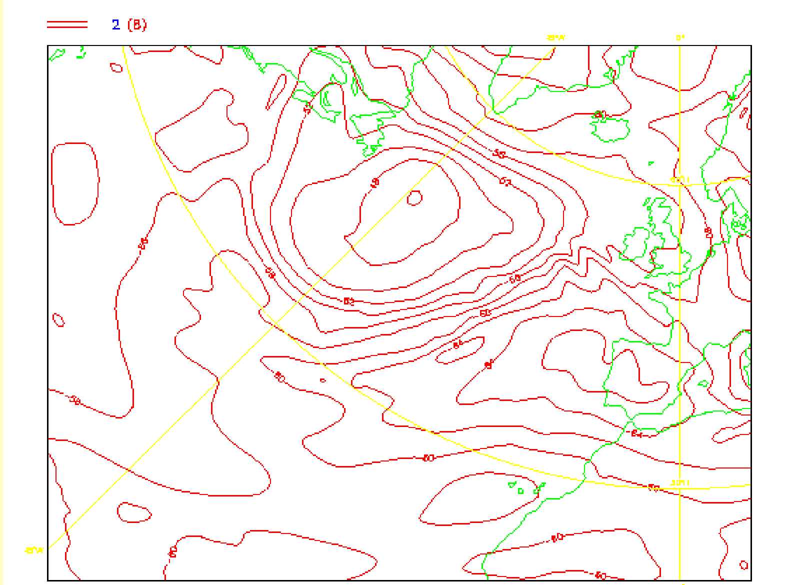
Stable extrapolation two-time-level semi-Lagrangian (SETTLS):

$$\varphi^A(t+\Delta t) \approx \varphi^D(t) + \Delta t \cdot \left(\frac{d\varphi}{dt}\right)_t^D + \frac{(\Delta t)^2}{2} \cdot \left(\frac{d^2\varphi}{dt^2}\right)_{AV}$$

where $\left(\frac{d\varphi}{dt}\right)_t^D = R^D(t)$ and $\left(\frac{d^2\varphi}{dt^2}\right)_{AV} = \left(\frac{dR}{dt}\right)_{AV} \approx \frac{R^A(t) - R^D(t-\Delta t)}{\Delta t}$

$$\varphi^A(t+\Delta t) = \varphi^D(t) + \frac{\Delta t}{2} \cdot \left(R^A(t) + \{2R(t) - R(t-\Delta t)\}^D\right)$$

Forecast 200 hPa T
from 1997/01/04
using SETTLS



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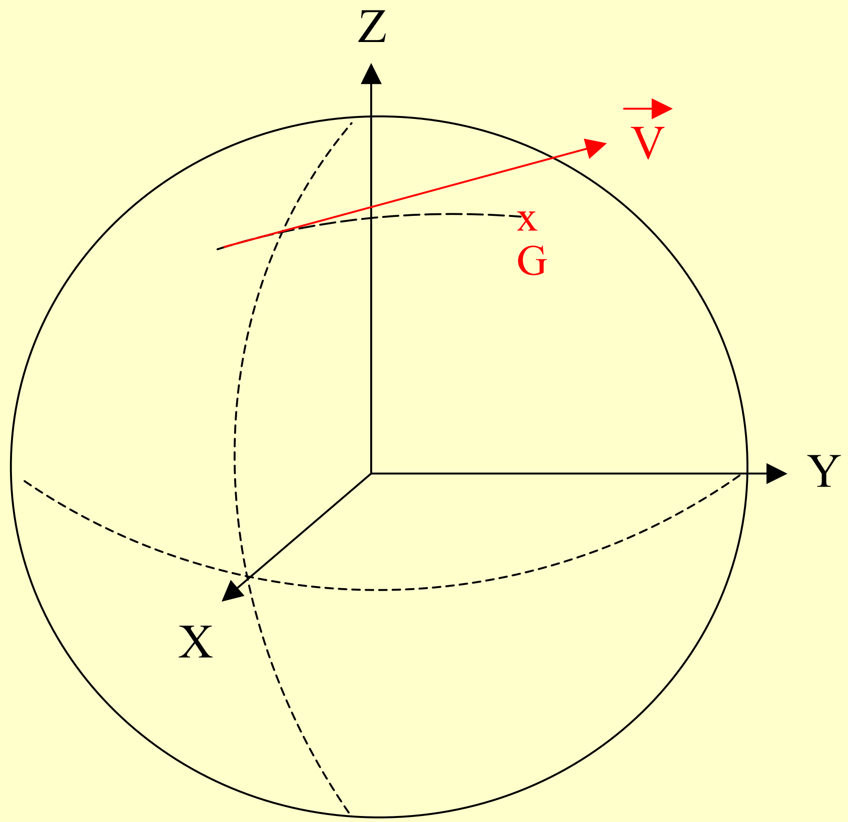
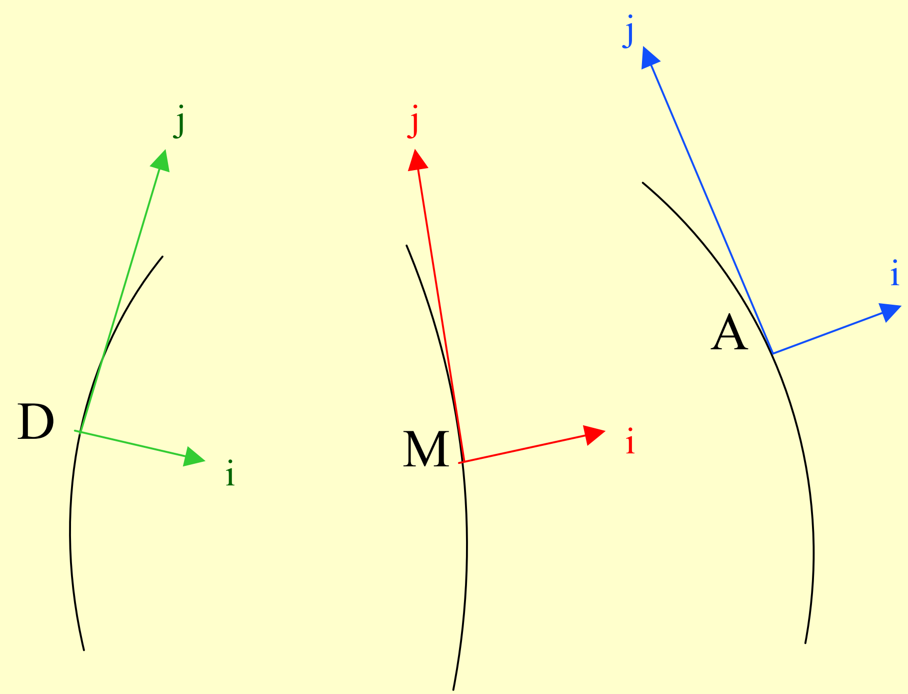
Spherical geometry in the semi-Lagrangian advection



Trajectory calculation

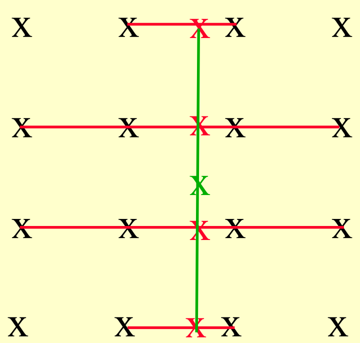


Tangent plane projection

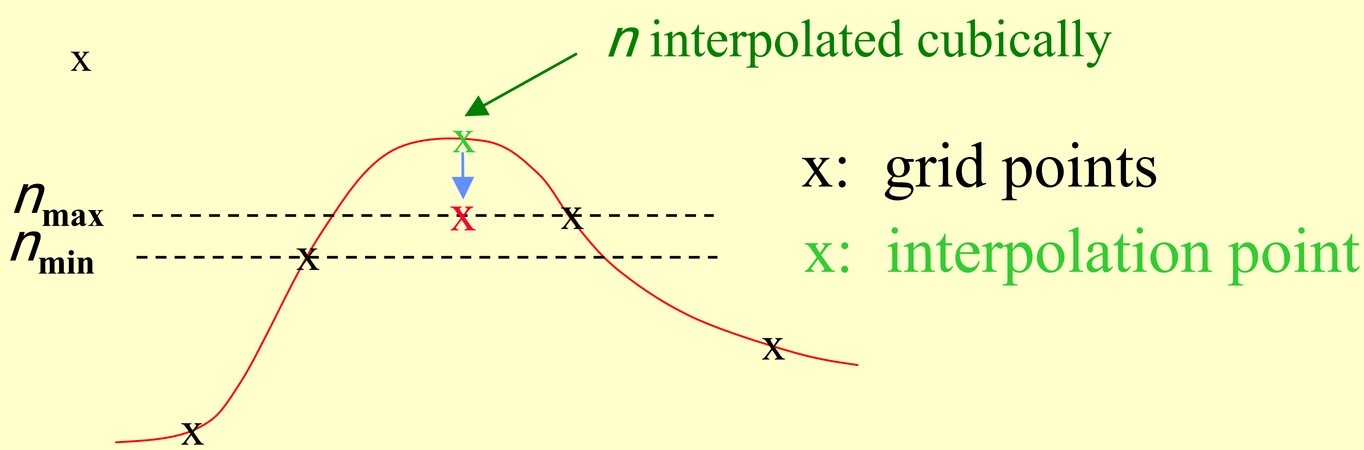
Interpolations in the semi-Lagrangian (1)

quasi-monotone Lagrange quasi-cubic interpolation

$$\varphi(x) = \sum_{i=1}^{4} C_i(x)\varphi_i \quad \text{with the weights} \quad C_i(x) = \frac{\prod_{k \neq i}^{4}(x - x_k)}{\prod_{k \neq i}^{4}(x_i - x_k)}$$



Quasi-monotone procedure:



Quasi-monotone interpolation is used in the horizontal for all variables and in the vertical for q and r_{O3}

Modified continuity & thermodynamic equations

Continuity equation

$$\frac{d}{dt}(\ln p_s) \equiv \frac{d}{dt}(l^* + l') = [RHS]$$

where $l^* = -\frac{\phi_s}{R_d \overline{T}}$ $\Rightarrow$

$$\frac{dl'}{dt} = [RHS] + \frac{1}{R_d \overline{T}} \vec{V}_h \cdot \nabla \phi_s$$

Reduces mass loss: D+10 $\Delta \overline{p}_s$ from 0.59 hPa to 0.02 hPa at T106L31

Thermodynamic equation

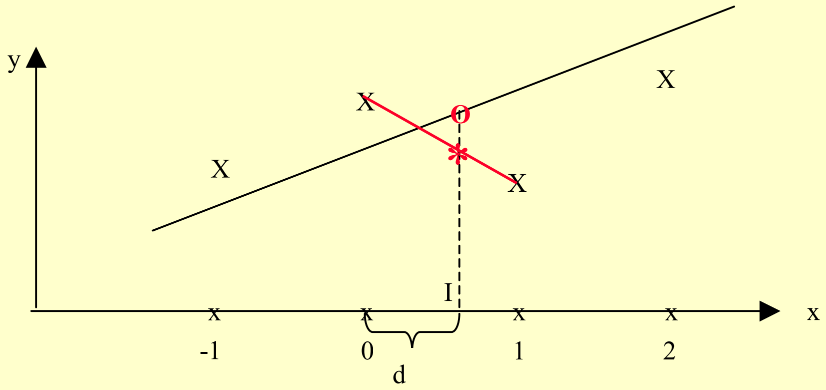
$$\frac{d(T - T_b)}{dt} = [RHS]_T - (\vec{V}_h \cdot \nabla T_b) - \dot{\eta} \frac{\partial T_b}{\partial \eta}$$

with $T_b = -\left(p_s \frac{\partial p}{\partial p_s} \frac{\partial T}{\partial p} \right)_{ref} \cdot \frac{\phi_s}{R_d \overline{T}}$

Reduces noise over orography.

Interpolations in the semi-Lagrangian (2)

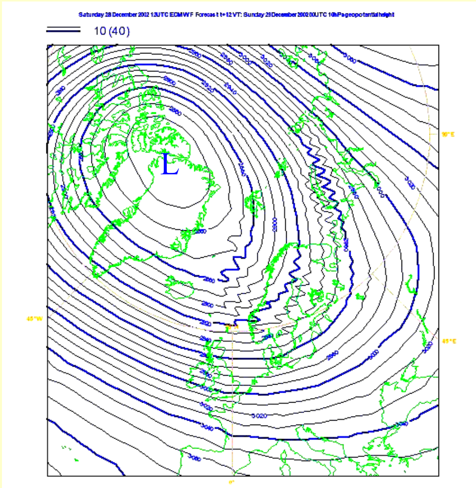
- Linear and smoothing interpolations:

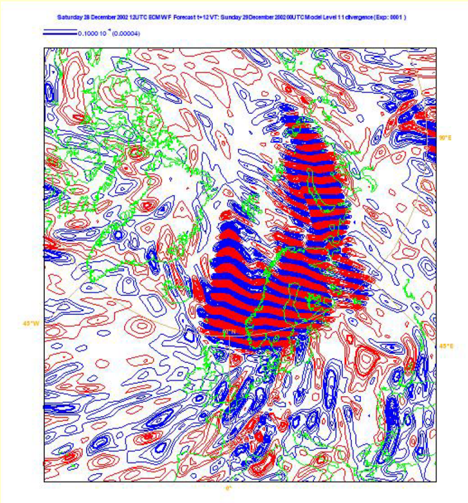


Linear interpolation is applied to the velocities needed in the trajectory computation and to the RHS of the forecast equations.
Smoothing interpolation is applied to the vertical velocity in the stratosphere.

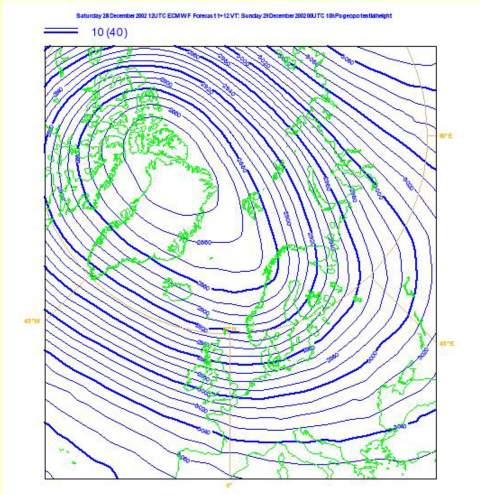
10 hPa Geopotential

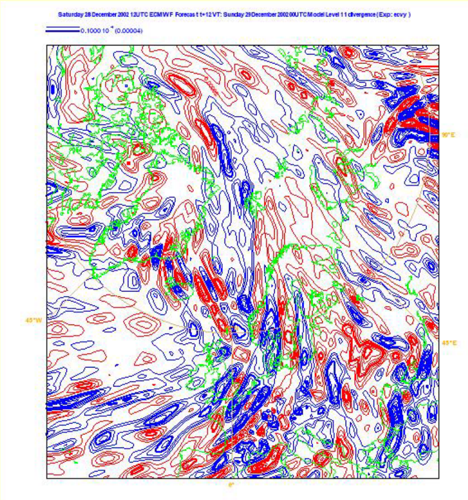
Divergence at model Level 11 (~5 hPa)





Linear interpolation in the computation of the semi-Lagrangian trajectory





As above but using smoothing interpolation for the vertical velocity

12 hour forecast from initial data of 2002-12-28 at 12

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Treatment of the Coriolis term

- Advective treatment:

$$f\vec{k} \times \vec{V}_h = 2\vec{\Omega} \times \frac{d\vec{R}}{dt} \Rightarrow \quad \frac{d\vec{V}_h}{dt} + f\vec{k} \times \vec{V}_h \rightarrow \frac{d}{dt}(\vec{V}_h + 2\vec{\Omega} \times \vec{R})$$

- Implicit treatment :

$$\frac{\vec{V}_h^+ - \vec{V}_h^0}{\Delta t} = -f\vec{k} \times 0.5(\vec{V}_h^+ + \vec{V}_h^0) + \ldots$$

Physical parameterizations

Coupled with the semi-Lagrangian scheme
(details in talk by A. Beljaars)

Horizontal diffusion

$$\frac{\partial X}{\partial t} = -K\nabla^4 X$$

- Implicit solution in spectral space

$$\frac{X_{n,m}(t+\Delta t) - X_{n,m}(t)}{\Delta t} = -K\nabla^4 X_{n,m}(t+\Delta t) = -K\left(\frac{n(n+1)}{a^2}\right)^2 X_{n,m}(t+\Delta t)$$

=>

$$X_{n,m}(t+\Delta t) = X_{n,m}(t)\frac{1}{1+K\Delta t\left(\frac{n(n+1)}{a^2}\right)^2}$$

- Analytical solution in spectral space

$$\frac{\partial X_{n,m}}{\partial t} = -K\left(\frac{n(n+1)}{a^2}\right)^2 X_{n,m} \Rightarrow \quad X_{n,m}(t+\Delta t) = X_{n,m}(t)e^{-K\Delta t\left(\frac{n(n+1)}{a^2}\right)^2}$$

Summary of the numerics in the ECMWF atmospheric model

- Two-time-level semi-Lagrangian advection
  - SETTLS scheme
  - Quasi-monotone quasi-cubic interpolation
  - Linear and smoothing interpolation for trajectories
  - Modified continuity & thermodynamic equations
- Semi-implicit treatment of linearized adjustment terms
- Cubic finite elements for the vertical integrals
- Spectral horizontal Helmholtz solver (and derivative comp.)
- Linear reduced Gaussian grid
- Semi-Lagrangian coupling of physics and dynamics

Future developments

- Non-hydrostatic version of the model
- Improve semi-Lagrangian interpolation
- Spectral representation by double Fourier series
- Improve conservation of advected quantities
- Study the influence of boundary conditions
  - for the semi-Lagrangian advection
  - for the vertical finite-element representation
- Investigate noise on vorticity over orography (aliasing?)

THANK YOU for your attention !