THE STRUCTURE OF THE LYAPUNOV VECTOR

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1 The universal instability mechanism of time dependent flows

Consider the non-autonomous linear dynamical system:

$$\frac{dx}{dt} = A(t)\, x \,. \tag{1}$$

Such a system describes the evolution of an initial perturbation to a time-dependent atmospheric flow, or alternatively the evolution of an ensemble of initial errors imposed on the configuration space trajectory of a forecast model. We will show that temporal variation of A in (1) with sufficient amplitude leads generically to asymptotic instability. In order to identify this generic instability of time dependent flows and understand how it is essentially different from asymptotic instability of autonomous dynamical systems we will require that at each instant A have neutral or damped spectrum. It is necessary for such a neutral or damped operator to be non-normal for even instantaneous perturbation growth to occur. However, while it is well known that non-normality can lead to episodic growth, it is another matter to sustain these instances of growth to produce asymptotic instability. Key to understanding the mechanism of generic asymptotic instability in time dependent operators is the observation that if the instantaneously evaluated operators do not commute with each other then there is no metric in which the time dependent operator is always normal. The instability that results from applying this sequence of non-normal operators to the solution vector can be shown to be independent of the choice of metric. The instability results from concatenating the finite growth achieved by the instantaneous operator while avoiding through time dependence the decay that would eventually occur if the stable operator were autonomous.

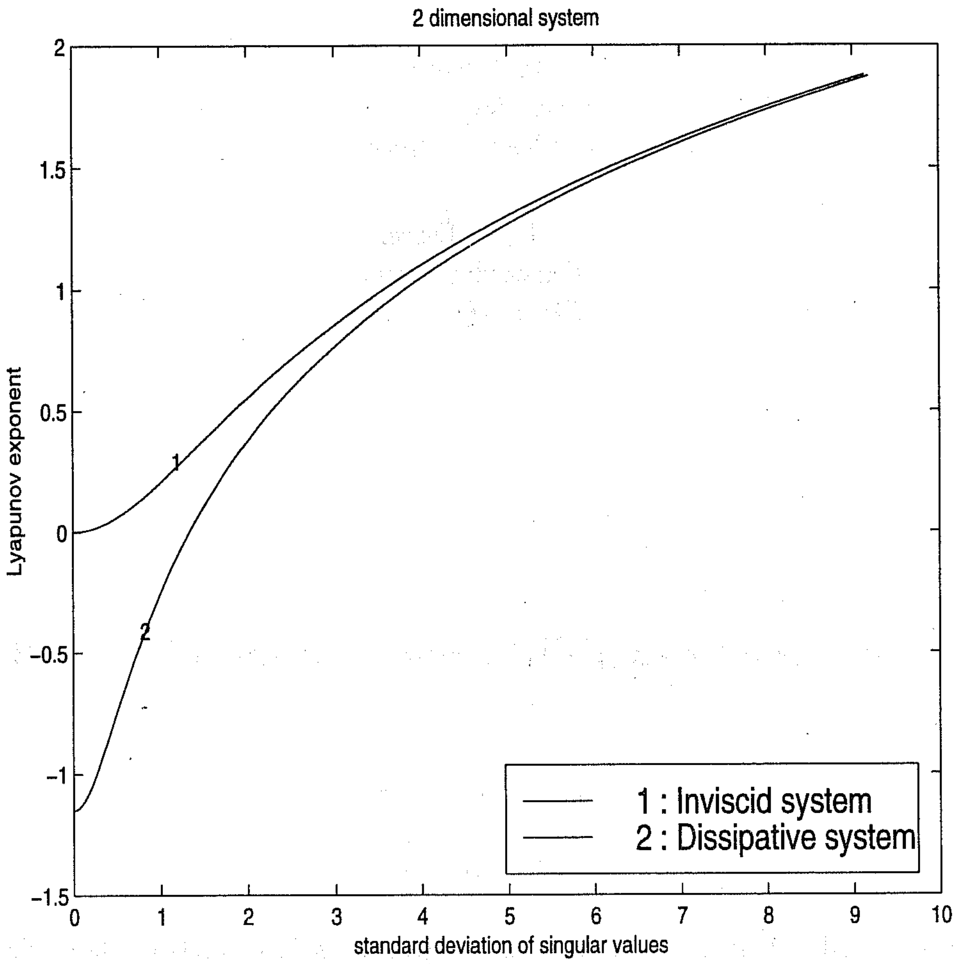

Figure 1: The Lyapunov exponent as a function of the standard deviation of the singular values of a two dimensional inviscid (curve 1) and a representative dissipative system (curve 2).

This process of destabilization by time dependence can be understood conceptually through an analysis method proposed by *Zel'dovich et al.* (1984) to explain the mean exponential increase in the length of material lines embedded in a random divergenceless flow. The analysis is simplified if the continuous operator $A(t)$ is approximated by a piecewise constant sequence of operators, A_i, each of which represents the continuous operator for a finite time interval τ. The initial state x_0 evolves to the following state at time τ:

$$x_n = \left(\prod_{i=1}^{n} e^{A_i \tau} \right) x_0 \, . \tag{2}$$

Passing to the limit of long time the Lyapunov exponent is given by:

$$\lambda = \lim_{n \to \infty} \ln \left(\frac{||e^{A_n \tau} x_{n-1}||}{||x_{n-1}||} \cdots \frac{||e^{A_1 \tau} x_0||}{||x_0||} \right) \Big/ n\tau \, . \tag{3}$$

Each element of the product is the incremental growth of the Lyapunov vector magnitude, G_i, over the interval τ. Consequently, the time average of the logarithm of the individual growths G_i approaches the first Lyapunov exponent as $n\tau \to \infty$. The total perturbation growth over $n\tau$ is given by

$$G_i = \sqrt{\sum_{k=1}^{n} \alpha_k^2 \, \sigma_k^2} \tag{4}$$

where the α_k are the projection coefficients of the unit vector lying in the direction of the state vector on the optimal (or singular) vectors of the incremental propagator $e^{A_i \tau}$ and the σ_k are the associated singular values. Consider a non-dissipative operator that preserves volume in state space so that $\prod_{k=1}^{n} \sigma_k = 1$. If we assume that the unit state vectors are uniformly distributed over the configuration space hypersphere then the Lyapunov exponent would be given by the average growth:

$$\lambda = \langle \ln G \rangle = \frac{\int_{||\alpha||=1} d^n\alpha \, \ln \left(\sqrt{\sum_{i=1}^{n} \alpha_i^2 \, \sigma_i^2} \right)}{\int_{||\alpha||=1} d^n\alpha} \, . \tag{5}$$

It can be shown that for all distributions of σ_i (so long as at least one $\sigma_i > 1$) the Lyapunov exponent $\lambda > 0$, so that the dynamical system is asymptotically unstable despite the fact that at each instant the system has zero net growth (e.g. consider a material line embedded in a divergenceless fluid). It is remarkable under this circumstance that growth is inevitable even

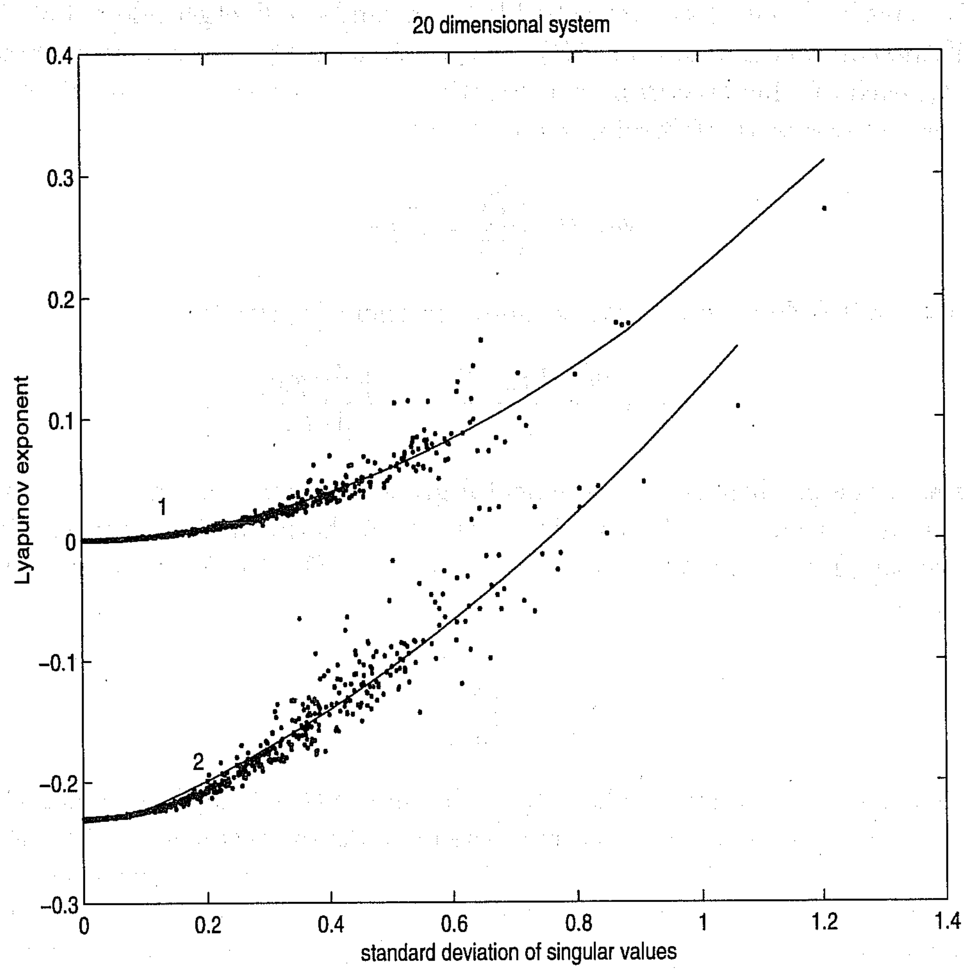

Figure 2: The Lyapunov exponent as a function of the standard deviation of the singular values of a twenty dimensional inviscid (curve 1) and a representative dissipative system (curve 2). The spread in the values is due to the fact that the singular values in a high dimensional system are not characterized solely by their standard deviation. The continuous lines provide a quadratic fit to the resulting Lyapunov exponents.

when the state vector projects uniformly on the optimal vectors of the instantaneous operator (the two-dimensional case is discussed at length in *Farrell and Ioannou* (1996)) . In practice for atmospheric applications expression (5) gives significantly more accurate estimates for the Lyapunov exponent when the observed statistical distribution of the state vector on the singular vectors is also taken into account by weighting the directions in (5) appropriately.

The magnitude of growth calculated in (5) can be shown to depend primarily on the dispersion in growth rates over the characteristic interval τ which can be conveniently measured by the standard deviation (std) of the singular values. For non-dissipative systems if the std is zero the Lyapunov exponent is also zero but as we have shown for any other value of the std of the singular values of the propagator the Lyapunov exponent is positive. Asymptotically as the variance increases the Lyapunov exponent is found to approach the logarithm of the std. The accuracy of this dependence of Lyapunov exponent on the logarithm of the std of the singular values can be seen in Fig. 1 for a system with two degrees of freedom, and in Fig. 2 for a system with 20 degrees of freedom. An example with dissipation is also included in these graphs. While for dissipative systems it is not necessary that sufficiently small singular value std's lead to positive Lyapunov exponent, the exponent for dissipative systems soon asymptotes to that obtained in the inviscid case, demonstrating that this generic mechanism leading to destabilization of time dependent systems is robust to the effects of dissipation.

We have assumed in the above that the operators are piecewise constant for a period τ. It is of interest to determine the dependence of the Lyapunov exponent on τ. In the limit of small τ we have:

$$\lambda = \lim_{\tau \to 0} \left\langle \frac{\ln \|e^{A\tau}\|}{\tau} \right\rangle = \lim_{\tau \to 0} \left\langle \frac{\ln \|I + A\tau\|}{\tau} \right\rangle = \left\langle x^\dagger \frac{A + A^\dagger}{2} x \right\rangle , \tag{6}$$

and consequently if the Lyapunov vector is equally distributed in the space of the optimal vectors the Lyapunov exponent in the limit of small τ must approach $\lambda = \text{trace}(\langle A \rangle)$, which for an inviscid system is always zero. We conclude that if bounded fluctuations of the operator are rapid enough and uncorrelated then in the limit $\tau \to 0$ the Lyapunov exponent will vanish. For this reason white noise is an unphysical limit in that the amplitude of the fluctuations required to maintain fixed variance of the noise diverges as the noise correllation time is decreased. In the large time limit, i.e. for $\tau >> \tau_d$ where τ_d is the characteristic decay time of the norm of the instantaneous operators, the time dependent system approaches the autonomous system and we expect that the time dependent system will inherit the assumed Lyapunov stability of the autonomous system in this limit.

We anticipate that the Lyapunov exponent will peak at an intermediate τ and that this τ for which the maximum Lyapunov exponent is obtained depends on the time scale of maximum optimal growth.

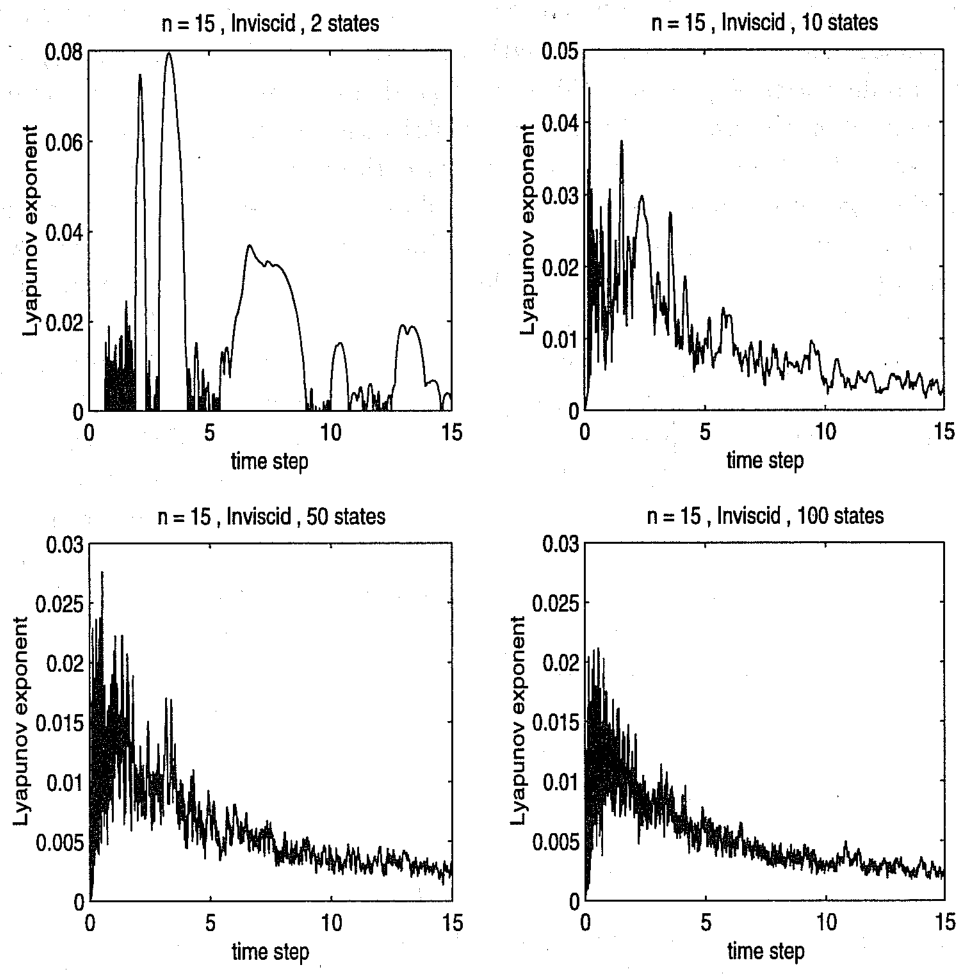

Figure 3: The Lyapunov exponent as a function of τ in an inviscid barotropic flow consisting of a periodic recurrence of two flow states (upper left corner), ten flow states (upper right corner), fifty flow states (lower left corner) and a hundred states (lower right corner). Although each state is neutrally stable and satisfies the Rayleigh necessary condition for stability, the composite time dependent state is unstable.

A simple model atmospheric system provides an example of the above results. Consider the evolution of harmonic perturbations to a time varying barotropic flow $U(y,t)$, where y is the northward direction. Choosing the inverse of the mean shear and the channel width as characteristic time and space scales respectively, the non-dimensional inviscid barotropic vorticity equation in a channel is given by:

$$\frac{\partial \nabla^2 \psi}{\partial t} = -ik\, U(y,t)\, \nabla^2 \psi - ik(\beta - \frac{d^2 U(y,t)}{dy^2})\, \psi \tag{7}$$

As before we consider flow states that are renewed every τ but obtain a further simplification by assuming that the sequence of these piecewise steady flow states recurs with period $n\tau$. The assumption of periodicity allows us to easily obtain the Lyapunov exponent by application of Floquet analysis. It is an assumption easily verified that as the period $n\tau$ is increased the growth of the first eigenmode approaches the growth of the Lyapunov vector. We consider periods of $n = 2, 10, 50, 100\tau$ with each flow state constrained to be neutral by imposition of the large non-dimensional $\beta = 10$ which renders the mean potential vorticity one-signed and assures stability of the individual state by the Rayleigh theorem. We have verified that the Lyapunov exponent for n large enough approaches the Lyapunov exponent of the non-periodic system and the approach to this limit is instructive. In the two flow state case (n=2) (upper left Fig.3) we obtain islands of instability characteristic of parametric instability in periodic systems such as the harmonic oscillator with periodic restoring force modulation the analysis of which results in the islands of instability of the familiar Mathieu equation solution. These islands of instability gradually blend into a continuum as the number of states increases ultimately producing the universal instability for all τ (cf. the lower right graph of Fig.3 for n=100). As discussed earlier it can be seen from Fig. 3 that the Lyapunov exponent vanishes in the limit $\tau \to 0$ and $\tau \to \infty$ and that the maximum Lyapunov exponent occurs at an intermediate time interval $\tau = O(1)$.

In order to sharpen the correspondence between our general analysis of instability in time dependent operators and the specific instability of the above atmospheric model we need to more closely relate the parameters in the analysis to the model. While the atmospheric flow does not evolve in piecewise constant steps nevertheless the time variation of the flow state is characterized by a finite decorrelation time of the order of a few days which we take to be the τ appropriate for our analysis. In addition to the time scale of jet vacillation we also know the mean state of the atmospheric jet and the spatial spectrum of variance about the mean state. Our model then consists in decomposing the flow state into a mean part and a time dependent part of stochastic nature which is adequately modeled for our purposes as a red noise process with the observed spatial structure, variance and decorrelation time.

Consider for illustrative purposes the evolution of errors in a zonally homogeneous barotropic channel with a Couette mean flow and with time dependent wind produced by modulation of

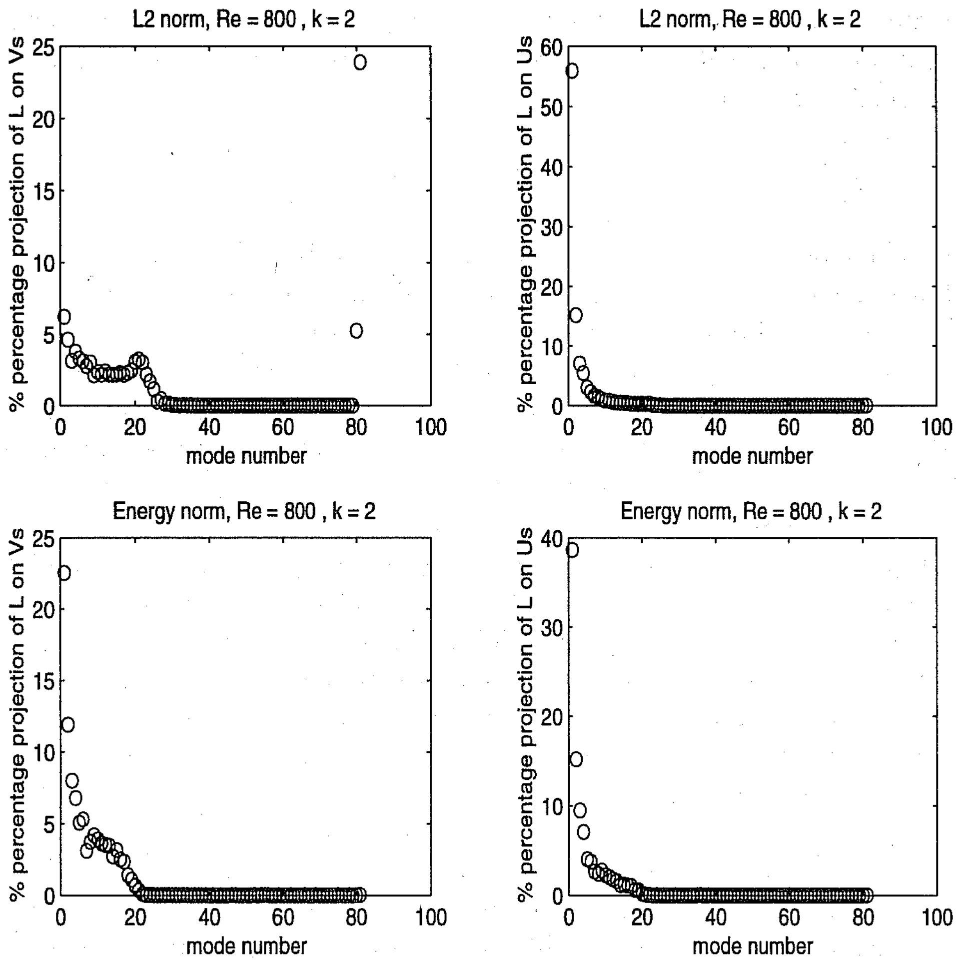

Figure 4: The mean projection of the Lyapunov vector on the optimal vectors (panels on the left) and the τ evolved optimal vectors (the panels on the right). The top panels are for L_2 metric, the panels on the bottom are for the energy metric.

the 5 gravest latitudinal harmonics. Forward integration yields the Lyapunov exponent which is positive even for $Re = 800$ with velocity fluctuations 10% r.m.s. of the mean. We have verified that the same Lyapunov exponent is obtained if the flow is approximated by a sequence of steady states each with time duration τ equal to the decorrelation time of the red noise process so that we can with confidence apply the theory developed above.

We turn now to the question of how the state vectors are distributed over the optimal vectors. Projection of the Lyapunov vector on the instantaneous optimal vectors and on the time evolved optimal vectors of the propagator over a period τ for the L_2 and energy metric is shown in Fig 4. We observe that the Lyapunov vector does not project uniformly on the optimal vectors in either of these norms. In the energy metric the Lyapunov vector projects entirely on the top 20 singular vectors. We can obtain an improved upper bound on the Lyapunov exponent if we take into account in the calculation (5) the probability distribution of the projection of the Lyapunov vector on the optimal vectors of the propagator over a decorrelation time. In fact a good approximation to the Lyapunov exponent is obtained by taking the mean square of the singular values in the evaluation of (5). This suggests that it is possible to obtain a good upper bound estimate for the Lyapunov exponent of the large scale atmospheric flow using the routinely collected analyses of singular value distributions. We observe that the Lyapunov vector projects strongly on the top evolved optimal vectors, and consequently the Lyapunov vector can be usefully characterized as having statistically the spatial structure of the top time evolved optimal vector.

2 Conclusions

We have demonstrated that the existence of a positive Lyapunov exponent in time dependent flows results from sustaining non-normal transient growth by a mechanism intrinsically related to the time dependence of the operator. While this instability mechanism is universal, it can be most easily analyzed in systems in which the Lyapunov vector projects equally on the optimal vectors of the propagator over the characteristic time τ of variation of the propagator. We have shown that if such a system is inviscid then the existence of even one growing direction is adequate for asymptotic instability. Considerations following from this analysis lead to estimates which can be applied to obtain the Lyapunov exponent of a forecast model and by extension of the atmosphere itself.

The Lyapunov vector is not a destabilized mode of the system, but is rather a vector lying in the non-normal subspace of the operator which exploits the transient growth resulting from interaction among the vectors in the non-normal subspace. We have found that in a barotropic channel flow the Lyapunov vector projects well on the first time evolved optimal vector which lies in this subspace.

3 References

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