PETROV GALERKIN FINITE ELEMENT SCHEMES FOR HORIZONTAL DISCRETIZATION AND CONSERVATION LAWS

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1. INTRODUCTION

Refined numerical methods are increasingly used in meteorological forecast models. The spectral method, used in the current ECMWF model, has a high numerical accuracy, resulting from a Galerkin method with spherical harmonic basis functions (see Machenhauer, 1979). The implementation of this method requires transformations from the spherical harmonic representation of a field to a grid point representation. This is done by a Fourier transform in the zonal direction and a Legendre transform in the meridional direction. Whilst fast Fourier transforms allow a very efficient treatment of the zonal direction, the Legendre transform programs applied presently are slow, requiring 0(N) operations per gridpoint, with N being the number of gridpoints in the meridional direction. Currently the ECMWF model maintains a reasonable balance between the numerical cost of transforms and other parts of the model. However, should progress in computer technology allow an increase of resolution in atmospheric models by an order of magnitude, an increasing part of the model's computational cost would be required by the Legendre transforms and alternative means of discretization are required. An estimate of the run time dependent on resolution was given by Steppeler (1987b).

Another reason to investigate alternative discretization schemes is the different ability of the schemes to represent local detail. All methods are able to represent features within their mesh length. However, the transport of features with sharp structures is often very inaccurate, involving strong deformation or dissipation of structures. The spectral method transports sufficiently smooth structures rather accurately, but it loses some resolution by the spectral transform. The model resolution is smaller than indicated by the transform grid, and the high order interpolation employed introduces oscillations of the fields which in many situations have to be considered as unphysical. Numerical methods which avoid the creation of artificial maxima and minima of fields should therefore be considered. This gives preference to a rather low order of interpolation of fields between the gridpoints, if sufficient accuracy can be maintained in this way.

Future model generations may incorporate the semi-Lagrangian method, as proposed by Bates et al. (1982) and Robert (1982). A numerical method should be able to support the Lagrangian method. A problem of the application of semi-Lagrangian methods to global models is that some of these don't formally conserve quantities like energy or mass. Structured elements can easily conserve mass, and, with more effort, may be able to conserve energy as well. Finally, discretizations of a local nature may offer advantages with respect to the computer organization of the code, depending on the computer architecture employed. Points to be observed are the use of memory, vectorization and multitasking.

The present paper will discuss the use of finite elements as an alternative discretization of the ECMWF model. These methods offer an economical way to obtain a high accuracy, and with respect to most operational features, like computation time per gridpint or organizational complexity, they perform rather like finite difference methods.

In meteorological applications normally finite elements on quasi regular grids are employed, as exemplified by Staniforth et al. (1977). More refined versions of the finite element method, as commonly applied in other fluid computational problems, are less popular for meteorological applications, though some limited research into their application to atmospheric and ocean simulation problems has been done. Crépon et al. (1984) used an unstructured irregular grid for an ocean model. Use of finite elements to achieve a regular resolution on the sphere was proposed by Baumgartner (1985). Triangular elements for the formulation of a shallow water equations model were employed by Navon (1987). Even though such methods may have a high potential of efficiency, the present paper will consider only finite elements on a quasi-regular grid, where the solution procedure factorizes and for this reason offers great numerical economy according to Staniforth et al. (1986). A number of options for the choice of basis functions will be considered in Section 2.

For finite difference equations it has turned out to be important to observe conservational constraints with the discretization (see Arakawa, 1966). For model equations employing a non-divergent velocity field, standard finite element schemes are automatically energy conserving (see Fix, 1975;

Jespersen, 1974). Energy conserving finite element schemes for the Boussinesq equations were introduced by Cliffe (1981) and Lee et al. (1982). For the divergent barotropic equations, standard finite element schemes will not lead to formally conserving schemes. Conservation can be obtained by using special versions of the Galerkin schemes according to Steppeler (1987). Similar principles to obtain conservation are used with a three dimensional model by Steppeler (1988). Section 2 will give examples of finite element schemes conserving energy and potential enstrophy. These schemes are independent of the choice of basis functions, and they work as well for linear and quadratic elements, or the Lagrangian elements introduced in Section 3.

2. ENERGY AND ENSTROPHY CONSERVING FINITE ELEMENT SCHEMES

A finite element discretization is based on the field representations

$$\tilde{\phi}\ (\underline{r}) = \sum_{\nu=1}^{N} \tilde{\phi}_\nu\ e_\nu\ (\underline{r}) \qquad (1)$$

Eq. (1) defines the space of the approximating functions $\tilde{\phi}(\underline{r})$. The basis functions e_ν being non zero only in the interval $(x_{\nu-1}, x_{\nu+1})$, may be piecewise linear or quadratic polynomials, the corresponding approximation spaces according to (1) being denoted as Ω_1 and Ω_2. Piecewise constant basis functions may be used by considering them as distributions according to Steppeler (1987). For a general field $\phi(\underline{r})$, not necessarily represented by (1), an operator G is defined which approximates ϕ by an element $\tilde{\phi}$ of the space of approximating functions according to (1).

$$\tilde{\phi} = G\ \phi, \qquad (2)$$

$$(G\phi)\ (\underline{r}) = \sum_{\nu} \tilde{\phi}_\nu\ e_\nu(\underline{r})$$

G is defined by

$$(e_\mu,\ G\phi) = (e_\mu,\ \phi),\ \text{with}\ \mu\ \varepsilon\ \{1,\ldots,N\} \qquad (3)$$

For two functions $a(\underline{r})$, $b(\underline{r})$ (a,b) is derived by

$$(a,b) = \int dxdy\ a(\underline{r})\ b(\underline{r})\ \omega(\underline{r}) \qquad (4)$$

According to (3) the scalar products of the function $\tilde{\phi}(\underline{r})$ are the same as those of the function $\phi(\underline{r})$ to be approximated. (3) is a system of N linear equations for the coefficients $\tilde{\phi}_\nu$ in (1), determining the element $\tilde{\phi}(\underline{r})$ of the approximating function.

In (4) ω is a positive weight function. Galerkin operators G can therefore be distinguished by the different spaces Ω, on which they project, and by different choices for the weight function ω. In the present paper we use only the space Ω_1 of piecewise linear functions and the space Ω_2 of piecewise quadratic functions. At element boundaries continuity of the basis functions is required with methods of order 1 and 2.

Among the different choices of the quadratic space Ω_2 we choose the one defined in Steppeler (1987), which has the property

$$\Omega_1^2 = \{u(\underline{r})\ v(\underline{r}),\ u,\ v\ \varepsilon\ \Omega_1\} \quad \Omega_2 \tag{5}$$

In the following, two Arakawa finite element schemes will be defined, conserving energy and potential enstrophy, respectively.

An energy conserving Galerkin finite element scheme for the shallow water equations is defined by

$$\begin{aligned} \dot{u} &= -G_1\,(u\,u_x + v\,u_y + H_x) \\ \dot{v} &= -G_1\,(u\,v_x + v\,v_y + H_y) \\ \dot{H} &= -G_2\,\left((u\,H)_x + (v\,H)_y\right) \end{aligned} \tag{6}$$

u and v are represented by the space Ω_1 (linear elements) and H is represented by Ω_2 (quadratic elements). G_1 will therefore project on Ω_1, and G_2 on Ω_2. The weight functions ω being associated with G_1 and G_2 according to (4) are $\omega(\underline{r}) = H(\underline{r})$ for G_1 and $\omega(\underline{r}) = 1$ for G_2.

The following Galerkin finite element scheme will conserve the enstrophy $Z = \int dxdy\ \eta^2 H$:

$$\begin{aligned}
\dot{u} &= G_2 \left(\eta\, \tilde{u} - \left(\tfrac{1}{2}\left((G_1 u)^2 + (G_1 v)^2\right) + H\right)_x\right) \\
\dot{v} &= G_2 \left(-\eta\, \tilde{v} - \left(\tfrac{1}{2}\left((G_1 u)^2 + (G_1 v)^2\right) + H\right)_y\right) \\
\dot{H} &= -\, G_2 \left(\tilde{u}_x + \tilde{v}_y\right) \\
\eta &= G_1 \left(\frac{v_x - u_y}{H}\right) \qquad \tilde{u} = G_1\, H\, u \qquad \tilde{v} = G_1\, H\, v
\end{aligned} \tag{7}$$

In (7), u, v, H are represented by the quadratic space Ω_2 ($G_2 \Rightarrow \Omega_2$), and G_1 projects on the space Ω_1 of linear functions. The proof of the conservation properties of the scheme (6) is given by Steppeler (1987), and can be obtained in a similar way for (7).

3. LAGRANGIAN ELEMENTS

3.1 The formalization of Lagrangian elements

Semi-Lagrangian methods approximate equations of the form

$$\frac{\partial}{\partial t}\,\phi\,(t,x) = -\,u\,(x)\;\phi_x\,(t,x) \tag{8}$$

by

$$\phi^{n+1}(x) = \ell\;\phi^n(x) = \phi^n(x - u^*(x)\,\Delta t) \tag{9}$$

with n being the time level. In (9) u* is interpolated from the u-velocity field. In the present paper we will not give specific examples, how to compute u* from u, but in this respect one can draw on experience with finite difference Lagrangian methods. We deal here only with a case of one space dimension x. In addition to the finite element approximation spaces Ω_1 and Ω_2, introduced in Section 2, we will consider the space Ω_0 generated by piecewise constant basis functions on the elements defined by the grid x_ν. The methods given in the following are valid for all the approximation spaces Ω_0, Ω_1, Ω_2.

Generalization to more than one space dimension is possible. This is straight forward for the Ω_0 methods, but more complicated for Ω_1 and Ω_2 methods.

Lagrangian methods, as currently used (Robert, 1982), don't imply formal mass conservation. Achieving conservation of mass with a Lagrangian finite element method will be the only conservation law dealt with in this section. In the framework of Galerkin schemes mass conservation is rather trivial, and for this reason the choice $\omega=1$ with (4) will be sufficient to obtain conservation. However, if more advanced methods are required, the methods described in Section 2 can be adapted to obtain conservation of other quantities as well. In a finite element framework, ϕ^{n+1}, ϕ^{n} and u* in (9) are defined for all x, and we assume they are represented in one of the finite element approximation spaces Ω_ν. The most natural Galerkin scheme based on (9) would be

$$\phi^{n+1}(x) = G\ \ell\ \phi^{n}(x) = G\ \phi^{n}(x-u^*(x)\Delta t) \qquad (10)$$

With G being a Galerkin operator.

In (10) we have to assume that $u^* \varepsilon \Omega_\nu$, with $\nu=1$ or $\nu=2$. If the basic representation for u is in Ω_0, the Galerkin operation $G_1 = \Omega_0 \rightarrow \Omega_1$ may be used to obtain a suitable u*.

Experience of Bates (1982) with finite difference Lagrangian schemes indicated that a high order interpolation between gridpoints is required to obtain good results. This would suggest in (10) to use approximation spaces of high order.

High order approximation spaces are used much in meteorological models, and Steppeler (1987a) used finite elements of order 2 for the vertical with a 3-dimensional model. These models have a high accuracy when applied to smooth data. If, however, the data have strong gradients, the high order interpolation can create artificial maxima and may create artificial negative values for quantities which according to their physical meaning should be positive. Conservation of high gradients and positivity of fields is a requirement raised mainly in connection with the moisture variable of the model.

In order to discuss conservation properties of the Finite element Lagrangian method, we consider the discretization of a variable ρ obeying the following

equation of motion:

$$\frac{\partial}{\partial t}\rho = -u\frac{\partial}{\partial x}\rho - \rho\frac{\partial u}{\partial x} \tag{11}$$

In (11) u is variable in time. We don't give the equation of motion for u, since it's discretization does not affect the conservation of mass.

There are many ways to base mass conserving Galerkin schemes for (11) on (10). We give here an example using a Galerkin adaptation of the semi-implicit Lagrangian scheme used by Bates (1988).

$$(\rho^{n+1} - G\,\ell\,\rho^{n})/\Delta t = -\frac{\bar{\rho}}{2} G\,(u_x^{n+1} + u_x^{*n}) - G((\ell\rho')\,u_x^{*n}) \tag{12}$$

In (12) n is the time level and ρ is a constant such that $\rho = \bar{\rho}+\rho'$. u^* is the interpolated velocity used in the definition (9) of ℓ, and G is the Galerkin operator defined in (2), (4) using the weight $\omega(x) = 1$.

To prove conservation of the mass M for (12) form

$$\begin{aligned} M^{n+1}-M^{n} &= \int \rho^{n+1}(x)\,dx - \int \rho^{n}(x)\,dx \\ &= \int G\,\ell\,\rho^{n}\,dx - \frac{\bar{\rho}}{2}\Delta t \int G(u_x^{n+1} + u_x^{*n}) \\ &\quad - \Delta t \int G((\ell\,\rho')\,u_x^{*n})\,dx - \int \rho^{n}\,dx \end{aligned} \tag{13}$$

Using (9), defining $x' = x - u^{*n}(x)\Delta t$ and using the fact that G conserves linear quantities, we obtain:

$$\begin{aligned} &M^{n+1} - M^{n} = \\ &\int dx\ \rho^{n}(x')\ (1-u_x^{*n}(x)\Delta t) \\ &- \int dx\ \rho(x) \end{aligned} \tag{14}$$

Transforming to the variable x' in the first integral of (14), we obtain

$$M^{n+1}-M^{n} = 0 \tag{15}$$

The definition of Lagrangian elements will make approximation (10) accurate even for low order approximation spaces Ω. The Lagrangian element approximation will have the following properties:

A mass conserving approximation for (11) is provided by (12) for ρ being in any of the approximation spaces Ω_0, Ω_1, Ω_2 and u* having appropriate boundary conditions and being approximated in Ω_1 or Ω_2.

- For ϕ in (1) being approximated in Ω_0 (piecewise constant basis functions) the scheme given in (10) conserves the positivity of the field, and does not create artificial maxima or minima.

- For the test problem

$$\frac{\partial \phi}{\partial t} = - u_o \frac{\partial \phi}{\partial x} \qquad (15)$$

(v(x) in const. in (8)) 10 provides the exact solution for all cases whose initial conditions can be presented in the basic grid x_ν. The basic feature of Lagrangian elements is a grid x'_ν introduced in addition to the basic grid x_ν. While x_ν is constant in time, the x'_ν grid is variable, and adapts to the solution at each timestep.

3.2 An example for the treatment of the Lagrangian node points

Since the derivation of variants of this method is rather straight forward, we describe here only the most simple method. The assumptions are:

- The x'_ν grid has the same number of gridpoints as the x_ν grid, and we can assume: $x'_\nu \; \varepsilon(x_\nu, x_{\nu+1})$.

- ϕ in (10) is approximated in the space Ω_0 (piecewise constant basis functions) and u* is given as a function in Ω_1.

The x'_ν develop in time according to the following rules:

1) At the initial time x'^0_ν is chosen at the lee side of the interval $(x_\nu, x_{\nu+1})$, that is

$$x'^0_\nu = x_\nu \text{ for } u^*_{\nu+1} + u^*_\nu > 0 \text{ and}$$
$$x'^0_\nu = x_{\nu+1} \text{ for } u^*_{\nu+1} + u^*_\nu < 0 \tag{16}$$

2) In order to obtain the time development of x'_ν we compute the grid $\tilde{x}_\nu$ as

$$\tilde{x}'^{n+1}_\nu = x'^n_\nu + u^{*n}(x'_\nu)\ \Delta t \tag{17}$$

If there is just one $\tilde{x}^{n+1}_\nu$ per grid interval $(x_\mu, x_{\mu+1})$, these are the x'^{n+1}_ν for the next timestep:

$$x'^{n+1}_\nu = \tilde{x}'^{n+1}_\nu \tag{18}$$

If there is an interval $(x_\mu, x_{\mu+1})$ without an $\tilde{x}'_\nu$, an $\tilde{x}'_\nu$ is created according to (16) which then becomes x'^{n+1}_ν. If there is an interval $(x_\mu, x_{\mu+1})$ containing more than one $\tilde{x}'_\nu$: $\tilde{x}'_\nu \ldots \tilde{x}'_{x+\alpha} \varepsilon (x_\mu, x_{\mu+1})$, all except one $\tilde{x}'_{\nu+\beta}$ are deleted. The $x'_{\nu+\beta}$ not to be deleted is determined by the relation

$$\left|\phi^n_+ (\tilde{x}'_{\nu+\beta}) - \phi^n_- (\tilde{x}'_{\nu+\beta})\right| \tag{19}$$
$$= \max_\alpha \left|\phi^n_+ (\tilde{x}'_{\nu+\alpha}) - \phi^n_- (\tilde{x}'_{\nu+\alpha})\right|$$

with

$$\phi^n_+ (x_\xi) = \lim_{\substack{x \to x_\xi \\ x > x_\xi}} \phi^n(x) \tag{20}$$

and

$$\phi^n_- (x_\xi) = \lim_{\substack{x \to x_\xi \\ x < x_\xi}} \phi^n(x)$$

The $\tilde{x}'_{\nu+\beta}$ determined by (19) is the x'^{n+1}_ν.

3) The x'^{n+1}_ν according to b) are resequenced in order that

$$x'^{n+1}_\nu \in (x_\nu, x_{\nu+1})$$

After the grid x'_ν has been determined according to 1), 2), 3), the timestep is performed according to (10). The Galerkin operator G in (10) is that belonging to the Ω_0 approximation space on the combined x_ν and x'^{n+1}_ν grid. Note that $L\,\phi^n(x)$ in (10) is defined in the combined grids $\tilde{x}'^{n+1}_\nu$ according to (17) and the grid $\approx^{n+1}_\nu$ defined by

$$\approx^{n+1}_\nu = x_\nu + v^{*n}(x_\nu)\,\Delta t \qquad (21)$$

3.3 The organization of the computer code

When using (10) with Lagrangian elements of order Ω_0 to approximate (8), the computation is organized in the following way:

1) The grids $\tilde{x}'^{n+1}_\nu$ and x'^{n+1}_ν are computed following 1), 2), 3) above.

2) The grid $\approx^{n+1}_\nu$ is determined according to (21).

3) For the intervals (x_ν, x'_ν) and $(x'_\nu, x_{\nu+1})$ the points $\tilde{x}'^{n+1}_\nu$ and $\approx^{n+1}_\nu$ belonging to these intervals are determined. $L\,\phi^n$ is defined in the combined $\tilde{x}'^{n+1}_\nu$ and $\approx^{n+1}_\nu$ grids. The values of $L\,\phi^n$ in the intervals of the combined $\tilde{x}'^{n+1}_\nu$ and $\approx^{n+1}_\nu$ grids have to be determined.

4) $\phi^{n+1}(x)$ is determined according to (10) in the combined x_ν and x'_ν grids. The procedure is illustrated in Fig. 1. Note that errors can be generated only for nonconstant velocities u(x) or irregular meshes x_ν. In these cases the error is a smoothing error. Rather accurate solutions for nonconstant advecting velocities can be obtained when using more points x'_ν than there are x_ν.

Because of the additional grid x'_ν the resources needed for the Lagrangian element method amount to a doubling of the resolution, and storage for the grid x'_ν must be provided, since this grid varies in time. However, in this double resolution grid it is a rather simple method.

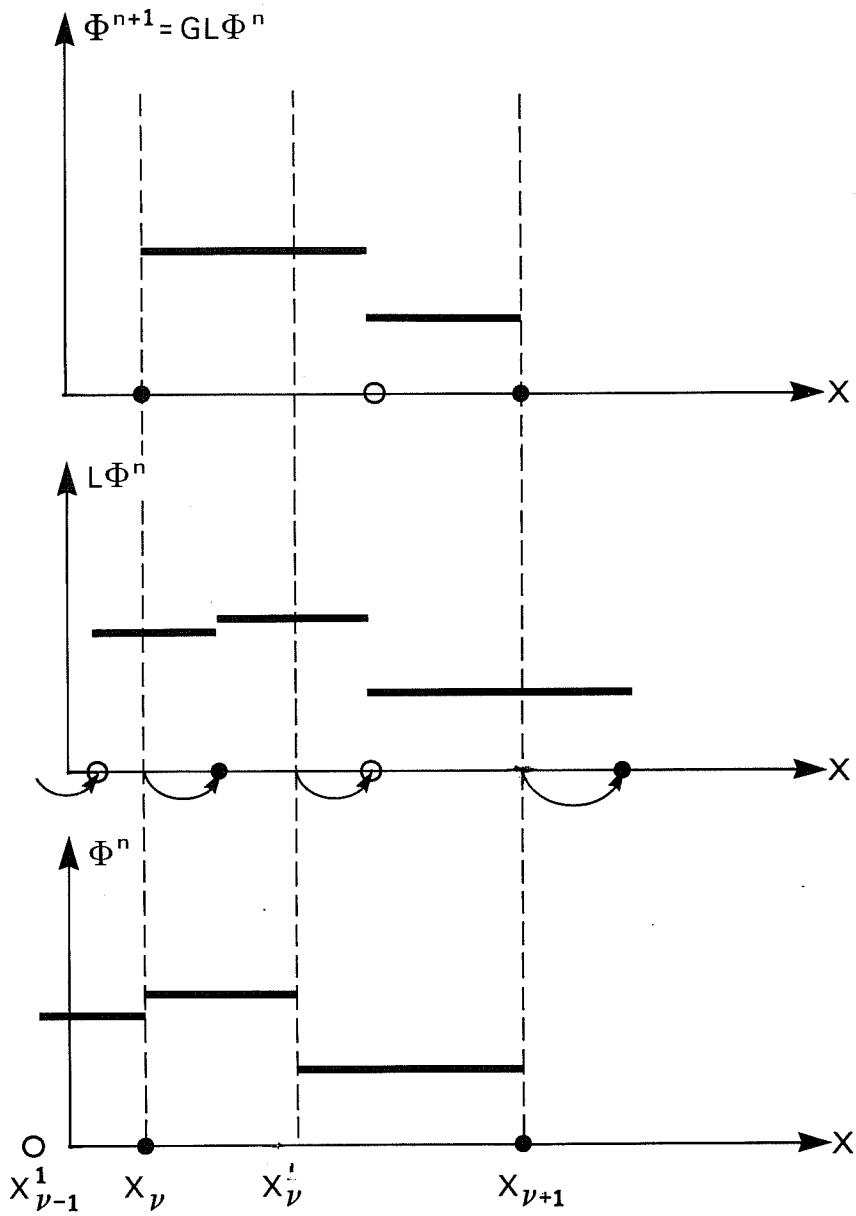

Fig. 1 A timestep with Lagrangian elements in the Ω_0 approximation space.

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